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MOD-7 TRANSFORMATIONS IN POST-FUNCTIONAL MUSIC

By

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ABSTRACT

Many musical compositions from the end of the nineteenth century and the beginning of the twentieth century retain some elements of functional tonality but abandon others. Most analytical methods are designed to address either tonal music or atonal music, but no single method completely illuminates this body of extended-tonal music. While both tonal and post-tonal theory have been extended in various ways to address this music, the use of tonal theory for analysis of this repertoire has not been completely formalized. The main obstacle for prolongational views of extended tonality is finding sufficient conditions for establishing that certain harmonies are structural in the absence of traditional harmonic function. In this regard, acoustical measures of stability, motivic connections, and chord equivalence all may form a part in determining the structural harmonies. Prolongational analyses of music may be represented by Schenkerian notation or transformational networks based on Lewin's *Generalized Musical Intervals and Transformations* (1987). This study explores a number of specific graphing techniques, including the diatonic lattice (Jones 2002), the just-intonation *Tonnetz*, and mod-12/mod-7 prolongational networks. After using group theory to explore the relationship of diatonic scale theory and tuning theory to transformational and prolongational analysis, excerpts from Wolf, Wagner, and Ravel are analyzed using mod-7 transformations. In giving support for prolongational analyses of chromatic and neo-tonal music, this study provides a case for tonality-based approaches to post-functional harmony.

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CHAPTER 1

CHROMATIC HARMONY, DIATONIC SCALE THEORY, AND POST-TONAL PROLONGATION

1.1 Introduction

Analytical approaches to tonality have generally attempted to provide some kind of internal justification and consistency in their applicability to all works in the genre. When applied to extended-tonal[1] and post-tonal musics, these approaches frequently fail to find such internal coherence.[2] Straus (1987) enumerates important criteria for finding defensible prolongations in non-tertian music.[3] This study addresses how one may distinguish structural from transient tones within a non-tertian harmonic language that clouds the difference between melodic and harmonic intervals. One aspect of the work toward a prolongational model of post-functional music will involve developing a structured approach to tonally interpreting post-tonal music that preserves at least harmonic function, parsimonious voice leading, or an underlying diatonic scale.

Much of the groundwork for such an approach has been laid by Jones (2002). According to Maisel (1999, 178), prolongation must be "organized around a single musical object—be it a chord, an interval, or a single pitch."[4] Further, "one must be able to show on the surface of the music how the listener could, in principle at least, cognitively organize the

[1]Here I use the term "extended tonality" in the same sense as Samson (1977, 151–153) to refer to music where sonorities are non-tertian or do not follow typical tonal harmonic function. This suppression of some aspects of traditional tonal hierarchies can be seen in passages from many nineteenth-century composers' music (Liszt, Wagner, Brahms, et al.) and in works by many twentieth-century composers (Scriabin, Schoenberg, Berg, Richard Strauss, Debussy, Prokofiev, et al.).

[2]Tonal approaches to this repertoire have frequently followed Schenker's analytical methodology or some reworking thereof. See Section 1.4 for a survey of such approaches. Other tonal theories commonly applied to chromatic music include neo-Riemannian and other function-theory approaches such as Harrison 1994. See Section 1.2 for a discussion of neo-Riemannian *Tonnetz* theory.

[3]Q.v. Section 1.4.

[4]Jones's approach suggests a view that prolongation can also connect two musical objects with the same tonal function. As the determination of function is more problematic in post-tonal music, Maisel's more conservative view may allow for more convincing readings.

intervening music so as to be able to connect distant points."[5] Jones (2002) provides a systematic means of finding salient contrapuntal connections between two sonorities using a property that he calls pervasive fluency (PF). Provided that a diatonic reading of a passage is possible,[6] pervasive fluency can offer the means of cognitively connecting two sonorities, regardless of whether they are tertian or whether the music follows traditional principles of functional harmonic progression.[7] The remaining problem is the need for criteria for establishing the sonorities that are to be heard as being structural. A possible solution for determining contextual stability relies upon whether the chords are referential. Several ways of establishing this quality deserve to be examined, including set-class equivalence and motivic association.[8] Further, we shall discuss the drawbacks of salience- and motive-based models and work toward a framework within which a prolongational span may be tested.

Provided the methodology and analytical notation necessary for effectively showing prolongational views of chromatic music and even of some post-tonal music, we can thus attempt to refine the existing tools for supporting hierarchical views of non-functional harmony and for aiding the analyst in making decisions on whether the music in fact maintains tonal paradigms. Thus, in addition to contributing to the existing analytical approaches for music on the fringes of tonal practice that view harmony from a tonal perspective, this project may also assist the analyst in the process of differentiating the distinctive properties of tonal and post-tonal musical languages. In this secondary purpose, I do not intend to make definitive judgments regarding the status of any music as belonging to the tonal or post-tonal repertoire. Instead, I hope that this study can give a clear sense of overlap between tonal (hierarchical) and post-tonal (associational) analytical methodologies for the music in question and that it can offer the possibility of fusing tonal and post-tonal analytical approaches in those works that feature characteristics of both harmonic languages.

The utility of this study lies in its applications in three areas. The first area of application is conceptual, offering a theoretical model supporting the claim that fundamentally tonal ways of hearing music can be informative in non-tertian and non-functional repertoires. Second, possibilities of musical interpretation based on the hierarchical structures and any

[5]This typically involves linear hearing of the intervening music. Jones's model derives from this contrapuntal practice of tonal hearing.

[6]In Chapter 4 we shall investigate what qualities distinguish diatonically unambiguous passages from blatantly post-tonal chromaticism.

[7]Jones's model in fact privileges tertian music, because of the properties of diatonic parsimony within the tertian system, as shown by Agmon (1991). Jones thus restricts his analytical purview to tertian harmony, albeit often highly chromatic and non-functional.

[8]Of course, prolongation need not be restricted to spans between sonorities of the same set class. A simple example of a prolongation connecting two distinct chord qualities is IV^6 passing through I^6_4 to ii^6_5. Prolongation may thus be asserted provided that the analyst can support a claim for the two sonorities' possessing the same harmonic function with linear motion connecting them. Asserting equivalence of function may prove a more difficult task in non-functional harmony. See footnote 4.

tonal ambiguities or conflicts can be suggested through the specific cases of the given analyses. Finally, cognitive structures elucidated by the theory can aid in decision-making with regard to issues of performance practice of the chromatic and post-tonal repertoire in question.

After surveying literature on diatonic scale theory, recent theories of chromatic harmony, and prolongational analysis of post-tonal works, in Chapters 2 and 3 we shall explore diatonic transformational models for analyzing music and discuss their theoretical basis and relationships. Then, Chapter 4 will focus specifically on the idea of prolongation in post-functional music, the context in which chordal stability and transience may be asserted, and the pitfalls of haphazard prolongational analysis. While the emphasis in this treatise is theoretical, examples of analysis using mod-12/mod-7 transformations will be used in Chapter 5 to show some areas of their potential use. The mod-12/mod-7 transformational model will first be used to show features of chromatic harmony and directional tonality. These examples will be drawn from Hugo Wolf's *Italianisches Liederbuch*. Non-functional progressions in the Prelude of Richard Wagner's *Tristan und Isolde* will also receive treatment using diatonic analysis. Finally, mod-12/7 transformations will be used to show the possibility of prolongation in non-tertian and non-functional portions of Maurice Ravel's *Valses Nobles et Sentimentales* and *Gaspard de la Nuit*.

1.2 Recent Theories of Nineteenth-Century Chromatic Music

Neo-Riemannian Theory

Neo-Riemannian theory centers around two spatial perspectives of chromatic harmony: the *Tonnetz* (tonal network), and transformational graphs first developed by Riemann (1880) and then formalized by Lewin (1987). The *Tonnetz* has appeared in various guises as a representation of pitch-class space that allows for the spatial mapping of harmonic motion. Morris (1998) calls these spatial maps "compositional spaces" and proposes similar types of graphs that focus on parsimonious stepwise connections among pitch classes called "voice-leading spaces." Cohn (1997) gives a brief history of the *Tonnetz*, which can be traced from its origins in Euler (1773) and its development by Oettingen (1866) through its significant use by Riemann (1915). Mooney (1996) gives a history of more recent work using the *Tonnetz*, and Hyer (1995) contributes significant refinements (e.g. 12-tone equal temperament applied to the *Tonnetz*) that were influential in its late-twentieth-century revival.[9]

[9] For older representations of tonal space (including the 3-limit space that we shall explore in Chapter 2), see Carey and Clampitt 1996a.

The *Tonnetz* offers at least two distinct ways of visualizing music meaningfully.[10] The first is the use of the just-intonation *Tonnetz* model to chart the motion of chromatic music through different key areas. The work of Harrison (2002) explores this area of research, especially in enharmonic progressions. The second use of the *Tonnetz* is as a spatial map of the voice leading between chords. With regard to equally-tempered views of the *Tonnetz*, much of the work of neo-Riemannian scholars[11] has centered around parsimonious voice leading, defining mathematical groups of voice-leading transformations that can be visualized on two- and three-dimensional *Tonnetze*. In Chapter 2, we shall examine several related mathematical groups and use them to build a set of relationships in tonal space that can be viewed on the *Tonnetz*.

Riemann (1880) is generally credited with the first creation of the types of diagrams of chordal relations that would evolve into transformational networks in Lewin's (1987) work.[12] Transformational graphs appear extensively in Lewin's subsequent work, in the neo-Riemannian issue of the *Journal of Music Theory* (42/2, 1998), and in a large amount of more recent scholarship. Hook (2002) provides a summary and bibliography of the literature, and contributes a unified system of transformations for triads. In Chapter 3, we shall consider a model for transformational networks that draws upon the features of Lewin's (1987) fundamental-bass networks and prolongational networks.

Diatonic Perspectives

Neo-Riemannian theory is designed specifically to describe those passages of chromatic harmony where tertian sonorities participate in non-functional progressions. It does not, however, provide the means by which one can incorporate the passages into a hierarchical analysis of the music. Neo-Riemannian analysis also does not address the relation of the non-functional passages with the diatonic scale that contributes to defining the tonality of the piece as a whole. The diatonic basis of these chromatic passages has been extensively debated in the literature. Proctor (1978, 149 ff) and McCreless (1983, 60–62) assert that when chromatic progressions transcend the established functional harmonic paradigms (diatonic root motion, diatonic resolution of unstable intervals, etc.), a diatonic harmonic foundation is no longer present. The neo-Riemannian theorists often take this perspective as well.

[10]A third use of the *Tonnetz* is found in the spatial representation of the ratios of just intonation and the relationship between just intonation and any practical tuning in common use. Because each new prime generator appended to the list of possible "consonances" adds another dimension to the tuning lattice, four- and five- dimensional constructions are possible and perhaps useful. This use of the *Tonnetz* will be addressed in detail in Chapter 3.

[11]E.g., Childs 1998, Douthett and Steinbach 1998, and Baker 2003

[12]See Klumpenhouwer 2002.

For example, Cohn (1998a, 171) addresses this issue when discussing Lewin 1987: "[T]he transformations in *GMIT* remain conceptually independent of diatonic tonality." On the other hand, Smith (1986, 109) and Harrison (1994) give arguments for the diatonic basis of harmonic function in chromaticism, and Brown (1986) presents Schenker's chart of *Stufen* (containing every chromatic degree except ♯IV/♭V) as a counterexample to McCreless and Proctor. More recently, Samarotto (2003) identified areas of conflict between the two views and provided a (tonal) conceptual model for the perception of the tonal conflicts in Brahms's most chromatic music. Jones (2002, 111) argues for the local diatonic interpretation of chordal successions, and builds upon Smith's view, asserting that a fundamentally diatonic distinction between stepwise motion (minor second) and chromatic inflection (augmented unison) is part of most trained listeners' perceptions of chromatic music. This study is based largely upon this perspective of chromaticism.

1.3 Diatonic Theory

Diatonic scale theorists speculate about the desirable properties of musical scales.[13] The diatonic/chromatic system used in tonal music is overdetermined. Several theoretically and historically important scales can be generated from cycles of fifths that are tempered by particular commas. The diatonic scale is the maximally even distribution of 7 notes within the 12-note chromatic scale. Likewise, the triad and seventh chord are maximally even distributions of 3 and 4 notes within the 7-note diatonic scale.[14] Because of the strong theoretical underpinning for the primacy of the triad and the 7-note diatonic scale established by theorists such as Agmon (1989, 1991), the theory has been extended into analysis of tonal music as the interaction of mod-12 and mod-7 systems by theorists such as Santa (1999) and Jones (2002).

Santa's dissertation "argues that the problems inherent in analyzing post-tonal diatonic music can be solved by a careful application of set theory modulo 7, in interaction with the more familiar mod-12 set theory." Along with the use of mod-7 set theory, Santa gives algorithmic procedures for gauging a passage's or work's centricity and a note's or chord's salience. In Santa's methodology for finding structural levels in diatonic post-tonal music, the finding of salient tones, traditional ornamental patterns, and motivic associations all contribute to the determination of structural and transient tones. While Santa's work arguably provides an associational theory of levels (following the recommendations of Straus

[13]Clough, Engebretsen, and Kochavi 1999, Carey 1998, Santa 1999, and Jones 2002 give complete bibliographies of the diatonic theory literature.

[14]Maximal evenness is defined by Clough and Douthett (1991).

(1987)) rather than a prolongational theory, when viewed from this perspective his analyses are informative and useful.

While Santa uses measurements of salience and motivic association to assert prolongation, Jones (2002) uses mod-12 and mod-7 systems to build upon more traditional notions of prolongational structure. Jones also provides a valuable way of visualizing Santa's "step classes."[15] Figure 1.1, which is based upon Jones's Example 2–2, p. 92, shows an example of the diatonic lattice. It represents the twelve chords between mm. 12 and 25 of Chopin's Scherzo, Op. 54. Vertically aligned mod-12 pitch-class integers on the lattice form chords (with A standing for pitch class 10, and B for pitch class 11). The placement of those 12-tone pc numbers on the 7 horizontal rows of the lattice (mod 7) indicate their relationships within the diatonic scale. It is thus a compelling method for visually rendering the interaction between the chromatic and diatonic realms. It may help to imagine the lattice as a cylinder where top and bottom levels are adjacent. If we rotate the cylinder so that it is seen from its edge, the mod-7 space becomes a kind of clock face with 7 hours on it. Horizontal and vertical lines are added around each pitch-class integer in the lattice in order to make it easy to read. The vertical lines on the lattice connect the members of each chord, and horizontal lines anticipate the "height" of each new tone relative to the diatonic positions of the members of the previous chord.

```
  |  |  |  |--3--2  |--3--3  |--3  |
  |--1  |--1  |  |--1--1  |--1  |  |
--B  |--B  |--B--B  |  |--0  |--B--B
  |--A  |--A  |  |--9--9  |  |  |  |
--8  |--8--7  |--8  |  |--8--8  |--8
  |--6  |  |--6  |--6--6  |  |--6  |
--4--4--4--4  |--5  |  |  |--4  |--4
```

Figure 1.1: Diatonic Lattice of Chopin, Scherzo, Op. 54, mm. 12–25

In addition to providing a compelling way of displaying diatonic space, the diatonic lattice also allows the analyst to visualize the diatonic contrapuntal lines that connect the structural chords in a prolongation. For this purpose Jones defines a property that he calls "pervasive fluency" (PF). When a chord progression can be heard as a harmonic prolongation, pervasive fluency offers a way of describing the contrapuntal pitch-class motions that create a sense of transience in the progression. Two types of primitive lines are permitted in pervasively fluent

[15] For an earlier invocation of musical representation using both mod-12 and mod-7 pitch-class integers see Brinkman 1986. Brinkman uses the term "name class" instead of Santa's "step class". In Chapter 2, I shall add a mod-3 term to Brinkman's ordered pair pitch-class representations to account for syntonic-comma differences on the *Tonnetz*.

passages (PFPs). A passing line may move only in a stepwise fashion from the initial chord to the final chord of the prolongation and may never reverse direction. It may also linger on any level of the lattice for any length of time. Similarly, a neighboring line may move only in a stepwise fashion and may linger for any stretch of time at one vertical position on the lattice. Any neighboring line, however, may only diverge vertically from its original location on the lattice by one diatonic step (either up or down), and must return to its initial level only once, and remain there until the final chord of the prolongation is reached. Once all primitive lines have been found between one pair of chords in the passage, we can determine whether it exhibits pervasive fluency by noting the starting and ending pitch classes of each line. A passage is pervasively fluent as long as a primitive line extends from every member of the first chord to some member of the last chord and from every member of the last chord back to some member of the first chord.

From this definition we can observe that the first five chords of Figure 1.1 form a PFP. Primitive lines extend from pitch class 4 in the first chord to pitch classes 3, and 6, and 11 in the fifth chord, from pitch class 8 in the first chord to pitch class 3 in the fifth chord, and from pitch class 11 in the first chord to pitch class 5 in the fifth chord. It would perhaps make more musical sense, however, to split this transient progression from a structural tonic harmony to a structural dominant harmony into two separate PFPs. First we would show the prolongation from the first chord to the third chord as neighbor motion ornamenting the tonic chord, and then from the third chord to the fifth chord as primarily passing motion with one chromatic lower neighbor embellishing $\hat{5}$.[16] Pervasive fluency provides the contrapuntal connections to support an abundance of potentially transient spans. In this case we were able to rely on traditional tonal chord function to decide which PFPs reveal a convincing prolongational structure. If the music expands traditional tonality in certain ways, however, pervasive fluency alone cannot help us decide what chords are structural and what chords are transient. Jones's model for describing the linear motion between prolonged harmonies nevertheless functions equally well in non-tertian music, once the structural harmonies have been chosen. In Chapter 4, we shall therefore focus primarily on methods for asserting which chords are structural, while acknowledging the necessity of a linear view of prolongation.

[16]This is only one possible interpretation of this interesting passage. Jones uses this example to discuss the voice-leading connections that support three possible interpretations of the first five chords and two possible prolongational views of the entire passage.

Just Intonation

In Chapter 2 we shall discover remarkable connections between diatonic scale theory and the traditional theory of just intonation (hereinafter JI).[17] These relations will be used as a basis for defining diatonic spelling and scale-degree function. So that we may discuss the theoretical application of JI later, I shall now define some of the main terms and notations of tuning theory. JI is the use of harmonic and arithmetic proportions in the construction of musical intervals.[18] According to current tuning theory, the goal of tuning according to whole-number frequency ratios is sensory consonance, embodied in beatless intervals and chords. Sensory consonance results when the single most significant beat pattern (the first- or second-order beat pattern) is eliminated (to the extent to which the ear is able to perceive it). Sensory consonance is not necessarily bound to frequency ratios of two small whole numbers. For example, an interval with the ratio 3002:2001 may be accepted by the ear as being "just". Nevertheless, the approximation of whole-number ratios in tuning provides a method for achieving sensory consonance.

JI restricts the ratios possible in the system using a prime limit. Any rational number $q = \frac{a}{b}$ in lowest terms, where a and $b \in \mathbb{Z}$, has prime limit p if and only if a's factors and b's factors $\subset \{\text{primes} \leq p\}$.[19] In other words, if a frequency ratio exists in JI of a particular prime limit, all prime factors of both the numerator and denominator of the frequency ratio (when expressed in lowest terms) are less than or equal to the prime limit. Thus, 3002:2001 does not exist in 5-limit JI, but 3:2 does. The traditional name for 3-limit JI is Pythagorean tuning, where all intervals are measured by the number of fifths (and octaves) that comprise them. 5-limit JI is called "just" or syntonic tuning, and adds pure major and minor thirds to the repertory of consonances. Finally, 7-limit tuning provides certain refinements to the tuning of seventh chords that are not part of the 5-limit system. The system of JI that I shall introduce in Chapter 2 is a 5-limit system, but 7-limit consonances can be successfully applied on top of the system to improve seventh chords.[20] Tonal music generally does not

[17] Just intonation is a theoretical system, and, while it certainly can serve as the basis for a practical system of adaptive tuning, its use here is purely in service to a theory of scale-degree function.

[18] The harmonic and arithmetic means were a central concept in music theory well into the eighteenth century. As tuning theory developed in the eighteenth and nineteenth centuries, the overtone series gradually became accepted as the scientific basis for the importance of these proportions. See Rasch 2002 and Barbour 1951 for the history of tuning and temperament, and specifically see Green 1969 for the development of harmonic-series theory. See also footnote 17 in Section 2.2.

[19] The mathematical symbols used here are defined in Appendix A.

[20] For example, the 7-limit interval 10:7 is less than 8 cents larger than the 5-limit diminished fifth 64:45, but may create seventh chords with less beating.

need higher prime limits to find consonant chords. Many microtonal composers nevertheless use higher-limit JI for interesting effects in their music.[21]

Although the use of JI is impractical or historically unjustified in the performance of much of the music in this study, JI nevertheless plays a theoretical role in diatonic scale theory as it relates to my work. The theoretical invocation of JI derives from the following rationales. First, the theory of JI provides a traditional theoretical perspective that is fundamentally tonal in its outlook on music, and thus can serve as a reference for making analytical decisions about music from the tonal perspective.[22] The tuning of chords in JI is based on the function each member of the chord serves in relation to the others.[23] More sophisticated notions of tonal function all rely on this more rudimentary type of function. Second, the relationship between JI and mod-7 diatonic theory suggests that the reconciliation of music to a mod-7 diatonic scale also represents a fundamentally tonal perspective.[24] In Section 2.3, I shall give the mathematical function by which all pitches in 5-limit JI can be mapped onto members of the 12-tone scale and 7-tone scale. This function forms a homomorphism from 5-limit JI to the mod-12 and mod-7 systems that are used by diatonic scale theorists and that are implicit in common musical notation. This provides a strong mathematical (and thus conceptual) connection between tonal music and the theoretical system of 5-limit JI, even when the music was never intended for performance in JI. The strictures for diatonic spelling in chromatic harmony that will be introduced in Chapter 2 thus derive from the principles of JI. I give the JI system used for deciding diatonic spelling in Section 2.1.

1.4 Prolongation in Post-Tonal Music

There is an extensive history of the application of theories of prolongation or structural levels to post-tonal music. Baker (1983) reviews much of the early literature. While Katz (1945) and Oster (1960) assert that the fundamentally tonal ways of elaborating background structures cannot be replicated in non-tonal contexts, several other theorists have made attempts to expand the prolongational perspective to encompass post-tonal idioms that may be seen as prolongations. Salzer (1952, 227) redefines tonality as “prolonged motion

[21] Among these composers are Harry Partch, Kyle Gann, James Tenney, and Ben Johnston. Compositions in higher-limit JI often require highly trained performers or computer performance because, other than in chords that follow the partials of the overtone series, 11- and 13-limit intervals are quite difficult to perform accurately without considerable experience in producing them. For more on designing notation systems for and traning performers to play in extended just intonation, see Johnston 1994.

[22] JI thus plays little role in the post-tonal analytical perspective that also may inform one’s hearing of the music.

[23] These functional relationships can be visualized as part of a tonal space that we shall explore in Chapter 3.

[24] The diatonic spelling of a note is also based on a rudimentary kind of tonal function.

within the framework of a single key-determining progression." To Salzer, "key-determining progressions" can be contrapuntal rather than functional. While Salzer conceptually expands the idea of tonality and what music may be addressed by Schenker's methods, he does not give any systematic means for interpreting the post-tonal structures that do not match functional harmonic procedures. Travis (1959, 1966, 1970) follows Salzer's lead, forging further into the post-tonal repertoire with the works of Schoenberg and Webern. Like Salzer, Travis does not define his method for discriminating the structural from the transient, relying instead on his ear and his intuition (and the notion that the beginning and ending sonorities tend to be heard as being more structural).

Morgan (1976) uses the idea that a dissonant sonority may be prolonged before resolving to create new middle and background structures for post-tonal works. As support for his claims, Morgan provides Schenker's analysis of a prolonged V^7 chord in J. S. Bach's C-major Prelude (WTC I, BWV 846) and Schenker's analysis (as a counterexample for "good" free composition) of a passage from Stravinsky's Concerto for piano and wind ensemble. As Salzer and Travis before him, Morgan shows how transient tones may be interpreted as contrapuntally elaborating structural chords without codifying how he decides what chords are structural or how to distinguish dissonance from consonance.

Some successful prolongational treatments of chromatic harmony include McCreless 1990, Darcy 1993, and Baker 2003. Lerdahl (1989, 73) proposes that a theory of atonal prolongation may be based on "salience conditions". Other scholars who have ventured into linear analysis of Schoenberg and Berg include Väisälä (1999) and Maisel (1999). Santa (1999, 2 (fn)), in addition to providing his own methodology for extending prolongational theory to diatonic post-tonal music, provides an extensive bibliography of more recent approaches. Among the sources that Santa cites, the most influential is Straus (1987). The success of any prolongational approach to post-tonal music since Straus's article depends in part on how it addresses his criteria for finding prolongation. These can be simply reduced down to one important consideration: Non-tertian music clouds the distinction between harmonic and melodic intervals. Because Schenker's theory depends upon this distinction for performing linear analysis, any expansion of this theory for non-tertian music must find alternative means for defining the ways in which transient tones elaborate upon structural chord tones to create a sense of prolongation.

Straus's theoretical proviso provides a worthy conservative basis for judging hierarchical theories of post-tonal music. In building a model for prolongational analysis of non-tertian music, one must thus establish how non-tertian chords may attain the status of structural harmonies. Provided an alternative means for deciding what chords—even non-tertian chords—are structural, contrapuntal lines passing between any two structural chords can still aid in hearing the passage prolongationally. In Chapter 4, I shall offer strategies for deciding what is structural in extended-tonal music and provide new theoretical qualifications that allow for a conservative evaluation of prolongational analyses.

CHAPTER 2

JUST INTONATION AS DIATONIC INTERPRETATION

In building a model for the transformational representation of tonal structures, it will be helpful to establish a comprehensive system of tonal relations. We can draw such a model of tonality from the theory of diatonic scales and the closely related field of tuning and just intonation (JI). JI has faced a great deal of controversy with regard to its practical applications, and it need not find practical use in any of the music that we shall explore in Chapter 5. It is my contention, nevertheless, that a theoretical JI system can aid in discovering scale-degree functions and tonal hierarchies. While in this chapter we shall be formulating a detailed system of 5-limit JI, and while this model for pure tuning can be adapted for practical use, our focus here will be on the theoretical aspects of tuning. This model of JI will be outlined in Section 2.1 in service of a larger theory of tonal relationships. Pitches in JI are tuned based on their functions within a scale and within the prevailing harmony, and these functions inform the diatonic spelling of these pitches and their placement within a tonal pitch space that can be displayed on a Cartesian plane.[1] The intimate relationship between diatonic scale theory and JI is explored in Sections 2.2, 2.3, and 2.4. This relationship will allow us to specify a mathematical function that can provide the spelling and tuning of any pitch or pitch class in a piece of music, given its mod-12 pitch-class integer and the key in which it functions. The coordinates of a pitch-class in tonal space can then be graphed on the just-intonation *Tonnetz* or in transformation networks, the properties of which are fully explored in Chapter 3.

[1]To be exact, the Cartesian-plane tuning lattice is a pitch-class space, while a three-dimensional graph is required for exact pitch representation. Both coordinate systems are discussed in this chapter, and Chapter 3 explicates the use of two-dimensional pitch-class lattices.

2.1 Tuning in 5-Limit Just Intonation

Just intonation (JI) has historically been portrayed as a tuning system where the chords are tuned properly at the expense of some melodic intervals, and often at the expense of the pitch center as well.[2] While I agree with the scholars who contend that JI should hold melodic intervals to a lesser standard than harmonically pure chords, I also believe that the tonic pitch should ideally be stable. Ear training and sight singing are typically taught based on scales and tonal relationships between learned scale-steps and chords (solfege or scale-degree singing). From this, one may conclude that the system of JI that most closely matches the pedagogical and theoretical tradition is scale-based. This type of JI can be made rigorous in the way that I shall show in this chapter, or it can be applied—less rigorously, but at least as effectively—to the practice of tuning voices and variable-pitch instruments in real-time performance. I believe that (after the first note) singers perform more accurately when they draw on their musical imagination for their tuning rather than drawing solely on already-sounding reference pitches.[3] Hence I propose the following system of JI.

Table 2.1 gives the scale that will be used for 5-limit JI.[4] All diatonic (and some chromatic) chords will use the scale degrees as spelled and tuned in the chart. The absence of $\sharp\hat{2}$, $\sharp\hat{5}$ and $\sharp\hat{6}$ in Table 2.1 suggests that, as long as the key is stable (i.e. there is no tonicization), modal mixture is the primary source of chromatic scale-degree inflection in tonal music.[5] Indeed, with the method of JI based on this scale system, all modulations and tonicizations (no matter how brief) require a change of scale.

[2] There has been a great deal of disagreement about this definition of JI. Further, it has been a common criticism of JI itself that pure tuning tends to result in a downward drift in pitch. See Klumpenhouwer 1992, Walker 1996, and Wibberley 2004 for three different views on this issue.

[3] The spirit of my perspective is in accordance with Eskelin 1994 and Mathieu 1997, although we disagree on certain important details.

[4] This view of the diatonic system displays a clear affinity to the "tonic" and "phonic" constructions found in Oettingen 1866. Oettingen thought of major and minor triads as inversionally related and expanded his notion of "tonicity" to include the entire major scale, and "phonicity" to include the inversion of the major scale (the Phrygian mode). One need not accept the notion of chord and key modality as contextual inversion to understand the polarity of the Ionian and Phrygian modes as a basic source for all modal mixture and scale-degree inflection within a key. For more background on harmonic dualism, see Harrison 1994 and Kopp 2002.

[5] For the sake of presenting all chords within a unified tonal space, all scale degrees and Roman numerals that appear within a theoretical discussion are given relative to the major mode. Musical examples in this dissertation, however, use conventional Roman numeral analysis relative to the prevailing key and mode.

Table 2.2 outlines the procedure of tuning music in this scale-based JI system.[6] The method that is outlined in the table formalizes the use of scale "mutation" to accomodate modulation and tonicization and also corrects some problems with the use of the scale in Table 2.1. First, some chords with chromatic alterations such as major III and minor ♭vi (chromatic mediants and double-mixture chords) are not spelled correctly when taken strictly from Table 2.1. Table 2.2 therefore offers a formal procedure for treating these situations. Rule 4 in the procedure also corrects an intonation problem in Table 2.1. Specifically, the supertonic triad derived from the scale in Table 2.1 contains a poorly-tuned fifth between $\hat{2}$ and $\hat{6}$ (one syntonic comma too narrow, at 40:27).[7] To correct this inconsistency in the scale, $\hat{2}$ can be relocated to 10:9 relative to tonic for as long as the ii chord is sounding. This may occasionally necessitate retuning a common tone between two chords (e.g. when the supertonic chord moves to a dominant chord). This theoretical retuning is to be avoided, if possible.

One way of avoiding this pitch shift is to tune the rest of the supertonic chord ($\hat{4}$ and $\hat{6}$) a syntonic comma higher, as long as this does not result in common-tone retunings as well. As this second solution cannot be consistently applied, the JI system outlined in Table 2.2 tunes $\hat{2}$ at 10:9 in all cases where neither $\hat{5}$ nor $\hat{7}$ is also present in the chord that contains $\hat{2}$. The advantage here is that, in typical situations of diatonic harmony, only $\hat{2}$ is ever inflected by a syntonic comma. In any other instance where this tuning procedure would have $\hat{2}$ at 10:9 the likelihood of common-tone retuning is even lower. While such an intonation shift is certainly undesirable, the distinction between the two tunings of $\hat{2}$ is not simply a byproduct of a dogged adherence to whole-number ratios. Indeed, the distinction in the two tunings of $\hat{2}$ maps on to the distinction between the two potential harmonic functions of $\hat{2}$—that is, as part of a dominant chord, or as part of a pre-dominant or subdominant chord. Through the examination of several musical examples that display peculiarities of scale-based JI we shall now begin to explore how differences in theoretical tuning map onto other theoretical distinctions that form part of our understanding of tonal harmony.

[6] Temperley 2000 inspired my own use the line of fifths in measuring tonal closeness. The line of fifths appears in Marx 1841, vol. 1. Enharmonic equivalence, of course, creates a circle of fifths, as given by Heinichen (1711). Following Weber (1832), one may combine the two ways of thinking into the idea of a spiral of fifths, where conceptual difference between enharmonically equivalent places on the spiral is maintained, but the leap from one note to its enharmonic equivalent is made easier by their radial proximity on the spiral. This metaphor agrees with the perspective shown by this JI system. Longuet-Higgins and Steedman (1971) also have developed a diatonic spelling method that begins with key determination by placing the pitch-classes of the melody on the just-intonation *Tonnetz*. Unfortunately, their algorithm is not comprehensive or consistent enough for our purposes.

[7] This problem has been known for centuries (see, for example, Mersenne 1637), and has been treated recently by Walker (1996). For a historical survey of this dilemma, see Rasch 2002. There is a second poorly tuned fifth in the scale in Table 2.1, namely ♭$\hat{3}$ to ♭$\hat{7}$. See the discussion of Figure 2.7 for the treatment of this problem.

Table 2.1: Preferred Diatonic Spellings/Tunings

$\hat{1}$	$\flat\hat{2}$	$\hat{2}$	$\flat\hat{3}$	$\hat{3}$	$\hat{4}$	$\sharp\hat{4}$	$\hat{5}$	$\flat\hat{6}$	$\hat{6}$	$\flat\hat{7}$	$\hat{7}$
1:1	16:15	9:8	6:5	5:4	4:3	45:32	3:2	8:5	5:3	16:9	15:8

Table 2.2: 5-Limit Just-Intonation Tuning Method

1. The scale degrees are to be spelled and tuned only as they appear in Table 2.1. If this results in a non-tertian spelling of a chord, use the procedure given in Rule 4 to decide the spelling of all members of the chord.

2. All dominant-function chords and tonicizations, including secondary dominants, secondary diminished seventh chords, altered dominants, and modulations, take the scale of the tonicized key as reference.

3. Common tones are never enharmonically respelled unless they are the result of a functional reinterpretation of a held interval between two chords (e.g. minor third becomes augmented second, or minor seventh becomes augmented sixth). This may result in progressions that wander diatonically (tonic becomes ♯VII or ♭♭II).

4. If the chord is not a dominant or leading-tone chord, use the scale built on the nearest note in the triad (not the seventh) to $\hat{1}$ and $\hat{5}$ on the infinite line of fifths. In other words, find the triad member that is closest to $\hat{1}$ and $\hat{5}$ on the line of fifths, and tune it according to the normal diatonic scale; then tune the rest of the chord according to the intervals of the scale that is based on that note.

Some exceptional situations within scale-based JI are somewhat more subtle than the syntonic-comma discrepency that we have just discussed. Because these too will have implications with regard to the analysis of non-tertian chords and various extensions of tonality, it will be worthwhile to work through each of these examples using the method given in Table 2.2. First, Figure 2.1 shows a progression displaying root motion by an equal division of the octave into three parts. In JI, of course, no interval divides the octave evenly, since no root of 2 is rational. One must therefore decide the diatonic spelling of the chords functionally. Rule 2 in Table 2.2 is not applicable in this case, as none of the chords has dominant function. For this progression, Rule 1 gives the spelling I III ♭VI I. Rule 4 helps in explaining this result, as the closest member of the III and ♭VI chords to $\hat{1}$ and $\hat{5}$ on the line of fifths are $\hat{3}$ and $\hat{1}$, respectively. Rule 3, however, dictates that no common-tone respellings—in this case, $\sharp\hat{5}/\flat\hat{6}$—are allowed when a change in the function of a held interval is not present. The progression must therefore be spelled as given in Figure 2.1 and consequently drift downward by a single diatonic step.[8] We shall return to the issue of diatonic drift in Chapter 3.

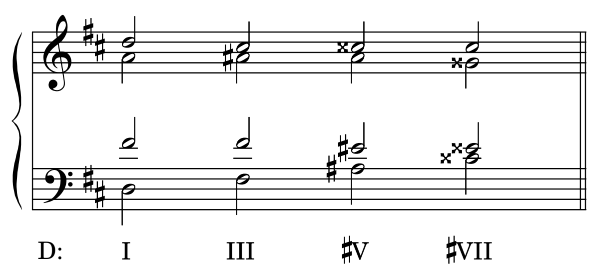


Figure 2.1: Example of an Enharmonic Progression

Figure 2.2 shows an alternation between tonic and a dominant-replacement chord that Cohn (1996) calls the hexatonic pole. Rules 1 and 4 give this progression the spelling shown in the example. In this case, $\flat\hat{3}$ is the closest member of the second chord to $\hat{1}$ and $\hat{5}$ on the line of fifths. This case, however, is somewhat ambiguous, as some may argue that ♭vi is an altered dominant chord. According to this argument, Rule 2 would give a non-tertian spelling of this chord with all chord members serving their dominant-function roles in the tonic key: $\hat{7}$ from V, $\flat\hat{3}$ from $\flat\mathrm{III}^{+6}$ and $\flat\hat{6}$ from $\mathrm{vii}^{\circ 7}$. If this chord is indeed intended as

[8] The term "diatonic drift" was coined by Jones (2002).

a dissonant entity, with functional resolutions of the scale degrees built in, then this far less consonant tuning of the chord is perhaps desirable. I am happy to allow the rules to possess this ambiguity in order to allow for case-by-case decision-making in chords such as the current example, the $\text{iv}^{\varnothing 7}$ chord, and other inflections of traditional dominant-function chords that appear in nineteenth-century chromatic harmony.[9] In Section 5.3 we shall pursue this ambiguity in the analysis of late-nineteenth-century chromatic harmony, where the two possible spellings will emphasize different aspects of the musical structure.[10]

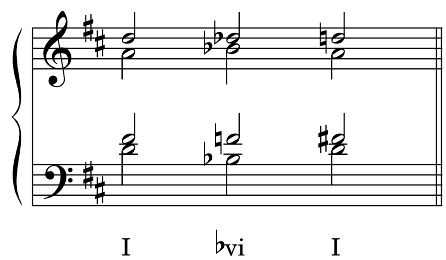


Figure 2.2: Example of a Chromatic-Neighbor Chord

Though the linear considerations of chromatic harmony sometimes prevail, there are also situations in chromatic music where the musical context resolves the multiple meaning. In Figure 2.3, the ♭vi chord appears again, this time not as a dominant-function harmony in the original key, but rather as a chromatic (sub)mediant participating in a falling-major-third progression. Once again, this progression results in diatonic drift, this time wandering up by a diatonic step, because of the overruling capabilities of Rule 3. In this case, since the chord can no longer be said to function as a dominant, we cannot justify a non-tertian spelling of the chord. The rules presented in Table 2.2 thus allow the analyst to make spelling decisions based on musical context. At the end of this chapter, we shall develop mathematical functions that give the same results with regard to diatonic spelling and theoretical tuning as the rules in Table 2.2. These functions, however, must still rely upon certain contingencies in Table 2.2, such as Rule 3. In Chapter 4 we shall explore further how certain musical

[9]Other chromatic chords, such as the common-tone diminished-seventh chord, the common-tone German-sixth chord, and more extravagant conglomerations of chromatic neighbor and passing tones, will strictly follow the spellings given by the key in which they are functioning.

[10]Smith (1986) explores similar multiple meanings in chromatic harmony and has influenced this study considerably.

contexts can clarify the diatonic spelling, harmonic function, fundamental bass note, or relative stability of a chord.

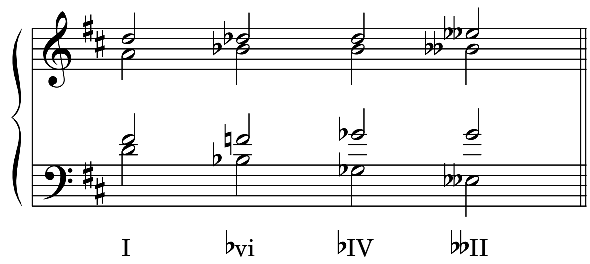


Figure 2.3: Musical Clarification of a Previously Ambiguous Spelling

In Figure 2.4, the III chord from Figure 2.3 appears now with a different function: It now acts as a secondary dominant. The choice in Rule 2 between a tonicization of vi and a tonicization of ♭♭vii should ideally follow the method in Rule 4 for establishing the closest possible relation to the original key. One would thus wish to spell the last two chords in this progression as V/vi and vi, in contradiction to the music notation in Figure 2.4. Even in this situation, however, Rule 3 overrides Rule 2 by maintaining the spelling of common tones. In this case, $\flat\hat{6}$ in the second chord cannot be respelled as $\sharp\hat{5}$ in the third chord. The progression here thus drifts upward by a diatonic step during the process of tonicization. As we shall discover in the next example, common tones may be respelled only in rare cases; and in such cases, a practical solution usually exists to avoid the concomitant intonation shift.

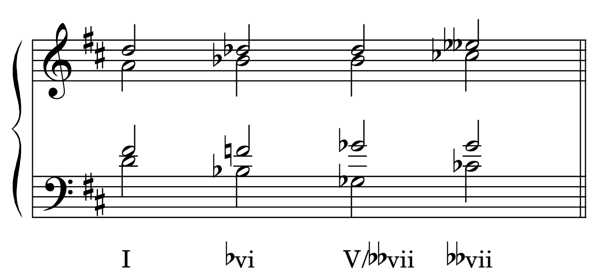


Figure 2.4: Example of Common-Tone Retention Forcing Diatonic Drift

By contrast, Figure 2.5 displays one example where Rule 3 allows common-tone respelling. Here, the function of the augmented second in the vii$^{\circ 7}$ chord as a dissonant interval is compromised by the passing chord in this diminished-seventh omnibus progression. The exception in Rule 3 applies only to this circumstance, where at least two common tones are held between two chords, and the function of the interval between the common tones changes from a consonance to a chordal dissonance, or vice versa.[11] The omnibus progression is one case where a relatively dissonant sonority—either a dominant-seventh or a diminished-seventh chord—is prolonged by a relatively consonant sonority—a major or minor triad. As such, some may embrace a misspelled triad between the two diminished sevenths in this progression, as the clear dissonance of the non-tertian passing chord would acoustically confirm its unstable passing function. In scale-based JI, there is no provision to disallow this reading of the progression. A similar progression, however, where the III chord is achieved by tonicization before resolving to the vii$^{\circ 7}$ – I that ends the present example, would certainly offer a clearer case for the use of Rule 3 to justify common-tone respelling.

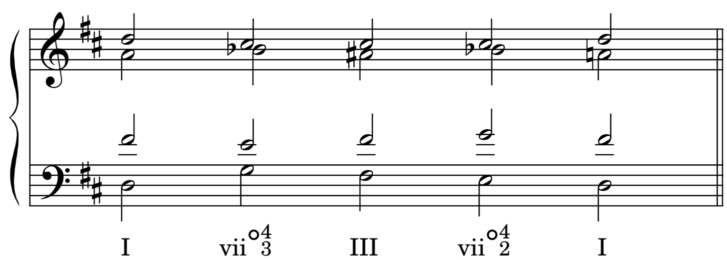


Figure 2.5: Example of Common-Tone Respelling

The remaining examples that we shall analyze are not concerned with diatonic spelling, but instead with notable matters of chord tuning.[12] Because the pitches of scale-based JI are tuned according to their harmonic function, the theoretical tuning of a pitch or chord can reveal its function and thus contribute to making decisions regarding a chord's fundamental bass and status within a non-traditional hierarchy of harmonies. The example shown in

[11]Non-chord tones do not merit enharmonic respelling, though they may occasionally be spelled in contradiction with the spelling guidelines to allow for stepwise neighbor motion.

[12]The distinctions we shall be making here will thus not be apparent in the musical notation of the examples. The differences in theoretical tuning nevertheless emphasize real differences treated by harmonic function theory.

Figure 2.6 contains two different tunings of half-diminished seventh chords, justified by their different tonal functions. First, vii^{ø7} has dominant function in the original key, and thus is tuned according to scale degrees $\hat{7}$, $\hat{2}$, $\hat{4}$, and $\hat{6}$ of that scale as 15:8, 9:8, 4:3, and 5:3, respectively. Calculated from these ratios, the intervals between adjacent chord members are 6:5, 32:27, and 5:4. The second half-diminished chord in the progression is ii^{ø7}, and follows Rule 4 for its tuning. Rule 4 forbids the use of the seventh in determining the chord member closest to $\hat{1}$ and $\hat{5}$ on the line of fifths. The closest scale degree to $\hat{1}$ and $\hat{5}$ in this chord, then, is not $\hat{1}$ (the seventh), but rather $\hat{4}$. In the key of the subdominant, the scale degrees of the chord, $\hat{6}$, $\hat{1}$, $\flat\hat{3}$, and $\hat{5}$, are tuned as 5:3, 1:1, 6:5, and 3:2. (In the original key, the ratios are 10:9, 4:3, 8:5, and 1:1.) The intervals within the chord are therefore 6:5, 6:5, and 5:4. This more consonant theoretical tuning of the half-diminished seventh thus serves its non-dominant function better than the more tense vii^{ø7} tuning seen in the second chord of the progression.[13] Further, the distinction between the $\hat{2}$ in the vii^{ø7}, at 9:8 relative to tonic, and the $\hat{2}$ in the ii^{ø7}, at 10:9 relative to tonic, highlights the two different functions of that scale degree can assume (dominant and subdominant).[14]

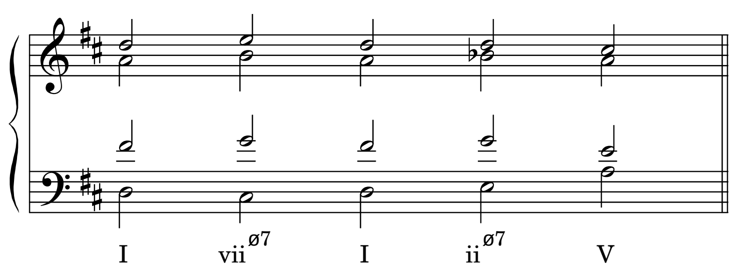


Figure 2.6: Two Distinct Tunings of the Half-Diminished Seventh Chord

The final example that we shall discuss, given in Figure 2.7, features two different tunings of the subtonic triad. The first half of the progression displays one possible harmonization of the descending-Phrygian-tetrachord bassline. When ♭VII is used as a diatonic chord in rare situations like this one, Rule 4 decides its tuning, with its root at 16:9 relative to tonic.[15]

[13] The diminished triad will thus also have two distinct tunings depending on its function as a pre-dominant (ii$^{\circ 6}$) or as a dominant (vii$^{\circ 6}$).

[14] This distinction reveals an important theoretical distinction between two instances of the same pitch class, but little or no practical importance in the approximation of pure tuning in performance situations. In most cases, performers do not make a conscious distinction between the different species of the whole step.

[15] This diatonic use of the subtonic is, of course, far more common in certain repertoires such as rock music.

According to Rule 4, ♭VII takes its tuning from the scale built on $\hat{4}$, and the third and fifth of the chord are tuned (in the original key) at 10:9 and 4:3, respectively. In other situations the chord typically functions as a dominant to ♭III (the relative major). Whenever ♭VII is followed by ♭III, Rule 2 decides the tuning of the chord as $\hat{5}$, $\hat{7}$, and $\hat{2}$ of the scale built on ♭III (6:5). The root of this chord lies at 9:5 relative to the original tonic, a syntonic comma higher than the diatonic ♭VII chord. Clearly the rest of the chord must also be a syntonic comma higher, with its third and fifth at 9:8 and 27:10, respectively. A similar situation obtains when the supertonic chord, which, according to Rule 4, usually lies at 10:9 relative to tonic, is replaced with a secondary dominant on the same root. The root and fifth of V/V are a syntonic comma higher than the root and fifth of ii. In situations where ii, through inflection of its third, becomes V/V, a strict reading of this tuning system would have $\hat{2}$ and $\hat{6}$ both shift up by a comma. Practical solutions, however, are usually possible that avoid any pitch shifts in held voices between two chords.

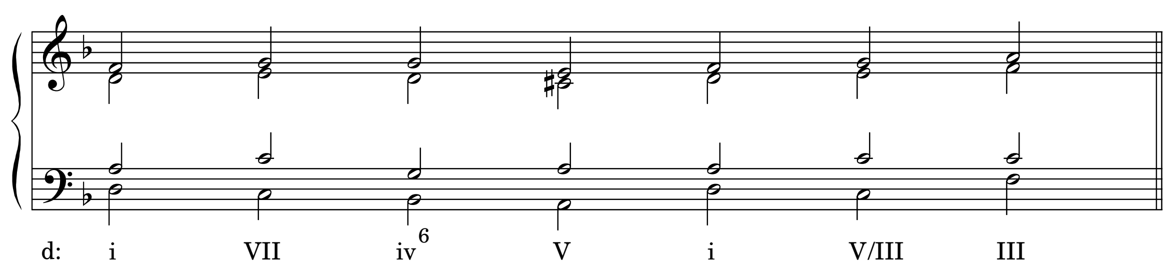


Figure 2.7: Example of Two Chords Distinguished by a Syntonic Comma

While Figures 2.1, 2.3, and 2.4 demonstrated progressions that exhibit diatonic drift, there are also progressions that exhibit syntonic drift, or motion away from tonic by a syntonic comma. Scale-based JI rectifies the syntonic drift of the traditional "comma pump" I vi ii V I; but it does not correct the drift of non-traditional progressions such as I (1:1) IV (4:3) ♭VII (16:9) ♭V (64:45) ♭III (32:27) I (80:81), where Rule 3 requires the maintaining of common tones regardless of syntonic drift. While practically it may be possible to apply a comma correction in an inconspicuous spot, strict scale-based JI will have to allow syntonic drift in such a case. The use of non-traditional progressions thus creates inconsistencies in tonal space which the analyst may choose to highlight if they seem to play a role in the music.

In Chapter 5 we shall examine how different ways of graphically displaying an analysis in tonal space can either emphasize or downplay unusual tonal features such as diatonic drift.

Whereas the rules in Table 2.2 are helpful with chromatic harmony, in typical tonal situations, however, the method can be applied intuitively based on the reference scale. Table 2.3 summarizes the tonal usage of scale-based JI. The first column of the table gives the intervals in abbreviated form, where numbers indicate the generic step size of the interval, and the letters d, m, M, and A symbolize the qualities diminished, minor, major, and augmented, respectively. The two sets of ordered triples given in the third and fourth columns of the chart will be discussed in the next section. The "usage" column in Table 2.3 shows examples of tonal situations in which each of the possible root intervals in scale-based JI occur. This column gives interval classes (abbreviated ic) for the typical intervals, indicating that, unless the tonal situation is among the exceptions listed at the bottom of the chart, all members of the interval class will employ the given ratio, its reciprocal, or either ratio multiplied by 2^x (where x is the number of octaves to be added to the simple interval). This chart should be used only as a quick reference, as it does not offer the kind of comprehensive guidelines for tuning that Table 2.2 gives. Scale-based JI combines procedures for deciding diatonic spelling and JI tuning into one comprehensive method. In the next section, we shall begin to explore this intimate relationship between tuning and diatonic spelling.

Table 2.3: Allowed Root Motion Intervals in 5-limit JI

Interval	Ratio	2,3,5 Powers	12,7,3 Steps	Usage
Typical:				
m2	16/15	$(4, -1, -1)$	$(1, 1, 0)$	ic1 (e.g. vii° – I, VI – V)
M2	9/8	$(-3, 2, 0)$	$(2, 1, 1)$	ic2 (e.g. IV – V)
m3	6/5	$(1, 1, -1)$	$(3, 2, 1)$	ic3 (e.g. vi – IV)
M3	5/4	$(-2, 0, 1)$	$(4, 2, 1)$	ic4
P4	4/3	$(2, -1, 0)$	$(5, 3, 1)$	ic5
A4	45/32	$(-5, 2, 1)$	$(6, 3, 2)$	ic6
Atypical:				
M2	10/9	$(1, -2, 1)$	$(2, 1, 0)$	V – vi or vi – V
m3	32/27	$(5, -3, 0)$	$(3, 2, 0)$	ii – vii°
Only when required by spelling rules:				
A1	25/24	$(-3, -1, 2)$	$(1, 0, 0)$	I – vii°/ii, etc.
d3	256/225	$(8, -2, -2)$	$(2, 2, 0)$	Aug.^{6} – vii°/V, ♭II – vii°
A2	75/64	$(-6, 1, 2)$	$(3, 1, 1)$	vii$^{\circ 7}$ – VI, etc.
d4	32/25	$(5, 0, -2)$	$(4, 3, 1)$	III$^+$ – vii°, etc.

2.2 Diatonic Spelling based on 5-Limit Just Intonation

The previous section provided spelling and tuning strictures for both harmonic and root motion intervals in 5-limit scale-based JI. The use of the strictures that were given in Table 2.2 results in the root motions given in Table 2.3. In this chart, two ways of expressing intervals as ordered triples are introduced. The first expresses the interval in 5-limit JI as powers of the first three prime numbers (2, 3, and 5).[16] For example, the interval 16/15 can be factorized as $2^4 \cdot 3^{-1} \cdot 5^{-1}$. The first, second, and third items in the ordered triple always refer to the powers of 2, 3, and 5, respectively. Therefore one need only write the exponents. In this case, 16/15 is written as $(4, -1, -1)$. This ordered triple can be thought of as the number of octaves, fifths, and major thirds that are used to create the interval. As 2, 3, and 5 are components of the harmonic series (all related to the fundamental), the perfect fifth represented by the power of 3 is actually an octave and a fifth, and the major third represented by the power of 5 is two octaves and a third.[17] At times it will be convenient to assume octave equivalence and only show the powers of 3 and 5—i.e. the second and third members of the ordered triple. We shall call the set of all such ordered triples $V_{2,3,5}$ and shall call the set of positive rational numbers with prime limit 5 that the ordered triples of $V_{2,3,5}$ represent N_5.[18]

The second ordered-triple notation for an interval is the number of steps the interval comprises in 12-tone, 7-tone, and 3-tone divisions of the octave. The first component of the ordered triple therefore gives the transposition in semitones. The second component indicates how many scale steps are involved in the interval (the T operation in mod-7 space). It would be logical to conclude that the third component indicates how many consonant skips along an arpeggio are involved in the interval, but the function of this third number is actually more complicated than this. I shall discuss the use of the 3-tone component of the ordered triple in Section 2.4. The first two components of the ordered triple can thus be translated into standard tonal interval names. To obtain the generic interval size, add one to the second number of the pair, so that zero will be a unison, one will be a second, two will be a third, and so on. The quality of the interval—major, minor, augmented, or diminished—is

[16] Oettingen (1866) is generally credited with the first use of this notation for the representation of intervals in JI. His preference, however, was to reverse the order, putting the power of 5 first and the power of 2 last. Klumpenhouwer (2002) provides a concise summary of Oettingen's notation and its use.

[17] The term harmonic series is used here to refer to the acoustical overtone pattern above a given fundamental pitch. The origin of the use of this mathematical term to refer to this acoustical phenomenon lies in the generation of the overtone pattern from a harmonic sequence of monochord divisions (string-length fractions $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$). Here the acoustical phenomenon of the harmonic series is represented by an arithmetic sequence (1, 2, 3, 4, 5, ...). The invocation of the harmonic series here is not to support claims for the acoustical privileging of the triad or any other musical structure. It does, however, form the acoustical basis for the definition of sensory consonance, as was given in Section 1.3.

[18] The mathematical relationships among the sets defined here will be formalized in Section 2.3.

encapsulated in the first number of the pair, but is dependent upon the second number. For example, an interval of a second that has one semitone is a minor second, two semitones is a major second, three semitones, an augmented second, and so forth. The first and second numbers in the ordered triple, then, are related to the mod-12 and mod-7 additive groups, respectively.[19] The mod-12 additive group, of course, is familiar from pitch-class set theory. A great deal of literature about diatonic scale theory discusses the tonal and mathematical relationships between 12-step and 7-step scales.[20] When the set of all ordered triples of integers is used to represent the 12-, 7-, and 3-tone representation of intervals, we shall refer to the set as $H_{12,7,3}$.

In order to relate these two ordered triples, we shall use them as row vectors.[21] The two 3×3 reciprocal matrices

$$H = \begin{bmatrix} 12 & 7 & 3 \\ 19 & 11 & 5 \\ 28 & 16 & 7 \end{bmatrix} \quad \text{and} \quad H^{-1} = \begin{bmatrix} -3 & -1 & 2 \\ 7 & 0 & -3 \\ -4 & 4 & -1 \end{bmatrix}$$

can be used to translate between these two interval notations using matrix multiplication.[22] Suppose that a and b are row vectors. The matrix multiplication operation $aH = b$ can be defined as

$$\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} h_{1,1} & h_{1,2} & h_{1,3} \\ h_{2,1} & h_{2,2} & h_{2,3} \\ h_{3,1} & h_{3,2} & h_{3,3} \end{bmatrix} = \begin{bmatrix} a_1h_{1,1}+a_2h_{2,1}+a_3h_{3,1} & a_1h_{1,2}+a_2h_{2,2}+a_3h_{3,2} & a_1h_{1,3}+a_2h_{2,3}+a_3h_{3,3} \end{bmatrix}.$$

Given the frequency ratio of an interval in JI, the conversion of the ordered triple using matrix H gives the correct diatonic spelling of that interval. The reverse is true as well, but to apply H^{-1} one must know the third term in the diatonic-spelling row vector. We shall

[19] While we have been implicitly using octave equivalence so far, the conversion matrices H and H^{-1} given here effectively translate *any* interval between its 5-limit JI representation and the number of half steps and scale steps in the interval. The relationships among both octave-equivalent and non-octave-equivalent JI and the diatonic spelling of notes in both pitch and pitch-class space will be made clear in Section 2.3.

[20] This binomial system of note or interval representation is discussed in more detail by Brinkman (1986). For more on the interaction of the diatonic and chromatic systems, see Agmon 1996a and Clough, Engebretsen, and Kochavi 1999.

[21] My use of row vectors and matrices here is based upon Karp 1984.

[22] We shall discover the significance of these two matrices and how they are constructed in Section 3.1. It is important that the two conversion matrices be reciprocal matrices in order for the map between $V_{2,3,5}$ and $H_{12,7,3}$ to be bijective.

return to this matter in Section 2.4, where we shall define the formula for generating the 3-tone component from the 12- and 7-tone interval sizes.

In scale-based JI, the same rules that decide diatonic spelling also decide tuning. This is because these spelling strictures derive from tonal function and the theory of JI. Despite Agmon's (1989, 1–2) claim "that the whole issue of intonation is for the most part irrelevant [to diatonic scale theory]",[23] this connection between diatonic spelling and JI is, in my view, essential to the theory of tonal and diatonic systems. Tuning in 5-limit JI is based on tonal function. Without tonal function, diatonic spelling is arbitrary.[24] Because diatonic spelling and tuning in JI are both based on tonal function, formal relationships can be expressed between certain mathematical groups that represent the frequency ratios of JI and the diatonic spellings of pitches. The next section explores this set of homomorphic groups that represent JI and the diatonic/chromatic system.

2.3 Homomorphisms Among Scale Systems and Tuning Systems

The previous section informally introduced two systems of representing intervals or pitches in 5-limit JI. We shall now formally relate the mathematical groups we have been using informally thus far. Mathematical formalisms are used here to clarify the relationship between diatonic spelling and JI that was discussed in the previous section. Readers who are less mathematically inclined may prefer to skip ahead to the concluding paragraphs of Section 2.4, as all that is practically necessary for successful analytical application of my theory is the decision-making apparatus given in Table 2.2.[25]

A pitch or interval in JI will be represented by a positive rational number q, where q is the frequency ratio of the two pitches in the interval, or the ratio of the pitch in relation to some other pitch (often the tonic). In section 1.3, the prime limit of the rational number q was defined as p, where $q = \frac{a}{b}$ in lowest terms, a and $b \in \mathbb{Z}$, and a's factors and b's factors $\subset \{\text{primes} \leq p\}$. Let N_3 be the set of all positive rationals q with prime limit 3 (Pythagorean tuning) and N_5 be the set of all positive rationals q with prime limit 5 (syntonic tuning). When we combine two intervals in JI, represented by two elements of N_3 or N_5, we use ordinary multiplication.

[23]To be fair, Agmon only claims that tuning is not relevant to his own model of the diatonic system, which forms a useful theory with significant influence on my own work.

[24]Without tonal function, spelling decisions would most likely be made only from convenience and ease of reading, rather than from theoretical considerations. Even in the notation of tonal music, ease of reading sometimes overrides theoretical correctness in diatonic spelling.

[25]For the less mathematically inclined reader I have included a glossary of mathematical terms and symbols in Appendix A.

Theorem 2.3.1 N_3 and N_5 form commutative groups under the operation of multiplication $q_1 q_2 = q_1 \cdot q_2$.

Proof 2.3.1 The set of rationals $\mathbb{Q}$ forms a commutative group under multiplication, and the set N_p of positive rationals with prime limit p is a subgroup of $\mathbb{Q}$. The identity element is 1, and the inverse of $q = \frac{a}{b}$ can be defined as $q^{-1} = \frac{b}{a}$. N_p thus contains the identity element of $\mathbb{Q}$ under multiplication. N_p is closed under multiplication because no two rational numbers with prime factors $\subset \{\mathbb{P} \leq p\}$ can ever form a rational number with any prime factor $\notin \{\mathbb{P} \leq p\}$. N_p clearly displays closure under under inverses because $\frac{1}{n}$ draws upon the same prime factors as n. (The reciprocal $\frac{1}{n}$ simply reverses the sign of the exponents on the prime factors of n.) ■

Figure 2.8 shows the relationships among N_5, N_3, and several other groups. The two other groups that are horizontally aligned with N_5 in the diagram are the two row vector notations for intervals in 5-limit JI, $V_{2,3,5}$ and $H_{12,7,3}$. Likewise, $V_{2,3}$ and $H_{12,7}$ are notations for 3-limit JI. Members of $V_{2,3,5}$ contain the powers of 2, 3, and 5 in the prime factorization of the ratio, and members of $H_{12,7,3}$ contain the number of 12-, 7-, and 3-tone scale steps that make up the interval. Thus, groups $V_{2,3,5}$ and $H_{12,7,3}$ both can be defined as the set of ordered triples (a, b, c), where $a \in \mathbb{Z}$, $b \in \mathbb{Z}$, and $c \in \mathbb{Z}$. Similarly, the groups $V_{2,3}$ and $H_{12,7}$ can be defined as the set of ordered pairs (a, b), where $a \in \mathbb{Z}$, and $b \in \mathbb{Z}$.

Theorem 2.3.2 $V_{2,3,5}$, $H_{12,7,3}$, $V_{2,3}$, and $H_{12,7}$ are all groups under the operation of componentwise addition $(a, b, c)(d, e, f) = (a + d, b + e, c + f)$. Moreover, $V_{2,3}$ and $H_{12,7}$ are subgroups of $V_{2,3,5}$ and $H_{12,7,3}$, respectively.

Proof 2.3.2 The set of integers $\mathbb{Z}$ forms a commutative group under addition. Because the set of possible values in each component of the ordered pair or triple is $\mathbb{Z}$, each component is a group. It follows that $V_{2,3,5}$ and $H_{12,7,3}$ are commutative groups under componentwise addition with identity $(0, 0, 0)$; and the inverse of (a, b, c) in either group is $(-a, -b, -c)$. It also follows that the subgroups $V_{2,3}$, and $H_{12,7}$ both have identity $(0, 0)$; and the inverse of (a, b) in either of the subgroups is $(-a, -b)$. Note that $V_{2,3}$ and $H_{12,7}$ behave like $V_{2,3,5}$ and $H_{12,7,3}$ (respectively) where the third component of the ordered triple is always 0. As 0 is the identity element of $\mathbb{Z}$ under addition, $V_{2,3}$ and $H_{12,7}$ are thus closed under addition and under inverses. ■

In the previous section, I introduced the two matrices

$$H = \begin{bmatrix} 12 & 7 & 3 \\ 19 & 11 & 5 \\ 28 & 16 & 7 \end{bmatrix} \quad \text{and} \quad H^{-1} = \begin{bmatrix} -3 & -1 & 2 \\ 7 & 0 & -3 \\ -4 & 4 & -1 \end{bmatrix}.$$

The operation of multiplying the row vector by the matrix H will serve as the map h between the two groups ($h : V_{2,3,5} \rightarrow H_{12,7,3}$). That is, for the ordered triples $a \in V_{2,3,5}$ and $b \in H_{12,7,3}$, $b = h(a) = a \cdot H$ and $a = h^{-1}(b) = b \cdot H^{-1}$. The double-headed arrows between the groups that are aligned on the same row of Figure 2.8 indicate that the groups are isomorphic—that is, all three groups encapsulate the same information about the musical relationships between pitches, though in somewhat different ways. The map $v_5 : N_5 \rightarrow V_{2,3,5}$ can be defined for $q \in N_5$ and $(a, b, c) \in V_{2,3,5}$ by factorizing q such that $q = 2^a \cdot 3^b \cdot 5^c$. Likewise, $v_3 : N_3 \rightarrow V_{2,3}$ maps $q \in N_3$ to $(a, b) \in V_{2,3}$ with the formula $q = 2^a \cdot 3^b$. Clearly these are bijective functions. Two 2×2 matrices

$$H_3 = \begin{bmatrix} 12 & 7 \\ 19 & 11 \end{bmatrix} \quad \text{and} \quad H_3^{-1} = \begin{bmatrix} -11 & 7 \\ 19 & -12 \end{bmatrix}$$

will serve as the map $h_3 : V_{2,3} \rightarrow H_{12,7}$ and its inverse. That is, for the ordered pairs $c \in V_{2,3}$ and $d \in H_{12,7}$, $d = h_3(c) = cH_3$ and $c = h_3^{-1}(d) = dH_3^{-1}$. In order to map any value in N_5 (5-limit JI) to its corresponding value in N_3 (Pythagorean tuning), we can define $\pi : N_5 \rightarrow N_3$ such that, for $q = 2^a \cdot 3^b \cdot 5^c \in N_5$, $\pi(q) = 2^{a-4c} \cdot 3^{b+4c}$. Note that this is a surjective relation—that is, many elements in N_5 map to a single element in N_3. The map $\rho : V_5 \rightarrow V_3$ follows from π. Specifically, for $(a, b, c) \in V_5$, $\rho(a, b, c) = (a - 4c, b + 4c)$. The map $\sigma : H_{12,7,3} \rightarrow H_{12,7}$ is by far the simplest: For $(d, e, f) \in H_{12,7}$, $\sigma(d, e, f) = (d, e)$.

As Figure 2.8 shows, all of the mappings just defined will remain the same in octave-reduced pitch-class space, which is represented by the groups O_5, $W_{2,3,5}$, $I_{12,7,3}$, O_3, $W_{2,3}$, and $I_{12,7}$. Let O_5 be the set of positive rationals greater than or equal to 1 and less than 2 with prime limit 5. Likewise, let O_3 be the set of positive rationals greater than or equal to 1 and less than 2 with prime limit 3.

Theorem 2.3.3 O_3 and O_5 form commutative groups under the operation $q_1 q_2 = q_1 \cdot q_2 \cdot 2^{-\lfloor \log_2(q_1 \cdot q_2) \rfloor}$. As such, they share the properties of (1) closure, (2) associativity, (3) identity, (4) inverse, and (5) commutativity.

Proof 2.3.3 For a and $b \in O_3$ or O_5, both $1 \leq a < 2$ and $1 \leq b < 2$.

(1) If the set is closed, then (a) $a \cdot b \cdot 2^{-\lfloor \log_2(a \cdot b) \rfloor} \in \mathbb{Q}$, and (b) $1 \leq a \cdot b \cdot 2^{-\lfloor \log_2(a \cdot b) \rfloor} < 2$.

(a) By definition, $\lfloor x \rfloor \in \mathbb{Z}$ for all x. Thus, $-\lfloor \log_2(a \cdot b) \rfloor \in \mathbb{Z}$. Because $2^{\lfloor x \rfloor} \in \mathbb{Z}$ for all $x \geq 0$, and $2^{\lfloor x \rfloor} \in \mathbb{Q}$ for all $x < 0$ (actually, for all x), we can thus assert that $a \cdot b \cdot 2^{-\lfloor \log_2(a \cdot b) \rfloor} \in \mathbb{Q}$, as desired.

(b) If $1 \leq a < 2$ and $1 \leq b < 2$, then $1 \leq a \cdot b < 4$.
For all $(a \cdot b) < 2$, $0 \leq \log_2(a \cdot b) < 1$, $-\lfloor \log_2(a \cdot b) \rfloor = 0$, and $2^{-\lfloor \log_2(a \cdot b) \rfloor} = 1$.
For all $(a \cdot b) \geq 2$, $1 \leq \log_2(a \cdot b) < 2$, $-\lfloor \log_2(a \cdot b) \rfloor = -1$, and $2^{-\lfloor \log_2(a \cdot b) \rfloor} = \frac{1}{2}$.
Therefore, for all $(a \cdot b) \in O_5, 1 \leq a \cdot b \cdot 2^{-\lfloor \log_2(a \cdot b) \rfloor} < 2$, as desired.

(2) The operation $ab = a \cdot b \cdot 2^{-\lfloor \log_2(a \cdot b) \rfloor}$ is associative if, for all a, b, and $c \in O_5$, $(ab)c = a(bc)$.

$$
\begin{aligned}
(ab)c &= a(bc) \\
\left(a \cdot b \cdot 2^{-\lfloor \log_2(a \cdot b) \rfloor}\right) \cdot c \cdot 2^{-\lfloor \log_2((a \cdot b \cdot 2^{-\lfloor \log_2(a \cdot b) \rfloor}) \cdot c) \rfloor} &= a \cdot \left(b \cdot c \cdot 2^{-\lfloor \log_2(b \cdot c) \rfloor}\right) \cdot 2^{-\lfloor \log_2(a \cdot (b \cdot c \cdot 2^{-\lfloor \log_2(b \cdot c) \rfloor})) \rfloor} \\
2^{-\lfloor \log_2(a \cdot b) \rfloor} \cdot 2^{-\lfloor \log_2((a \cdot b \cdot 2^{-\lfloor \log_2(a \cdot b) \rfloor}) \cdot c) \rfloor} &= 2^{-\lfloor \log_2(b \cdot c) \rfloor} \cdot 2^{-\lfloor \log_2(a \cdot (b \cdot c \cdot 2^{-\lfloor \log_2(b \cdot c) \rfloor})) \rfloor} \\
2^{-\lfloor \log_2(a \cdot b) \rfloor - \lfloor \log_2((a \cdot b \cdot 2^{-\lfloor \log_2(a \cdot b) \rfloor}) \cdot c) \rfloor} &= 2^{-\lfloor \log_2(b \cdot c) \rfloor - \lfloor \log_2(a \cdot (b \cdot c \cdot 2^{-\lfloor \log_2(b \cdot c) \rfloor})) \rfloor} \\
2^{-\lfloor \log_2(a \cdot b) \rfloor - \lfloor \log_2(a \cdot b \cdot c) + \log_2(2^{-\lfloor \log_2(a \cdot b) \rfloor}) \rfloor} &= 2^{-\lfloor \log_2(b \cdot c) \rfloor - \lfloor \log_2(a \cdot b \cdot c) + \log_2(2^{-\lfloor \log_2(b \cdot c) \rfloor}) \rfloor} \\
2^{-\lfloor \log_2(a \cdot b) \rfloor - \lfloor \log_2(a \cdot b \cdot c) - \lfloor \log_2(a \cdot b) \rfloor \rfloor} &= 2^{-\lfloor \log_2(b \cdot c) \rfloor - \lfloor \log_2(a \cdot b \cdot c) - \lfloor \log_2(b \cdot c) \rfloor \rfloor} \\
2^{-\lfloor \log_2(a \cdot b) \rfloor - \lfloor \log_2(a \cdot b \cdot c) \rfloor - \lfloor \log_2(a \cdot b) \rfloor} &= 2^{-\lfloor \log_2(b \cdot c) \rfloor - \lfloor \log_2(a \cdot b \cdot c) \rfloor - \lfloor \log_2(b \cdot c) \rfloor} \\
2^{-\lfloor \log_2(a \cdot b \cdot c) \rfloor} &= 2^{-\lfloor \log_2(a \cdot b \cdot c) \rfloor}
\end{aligned}
$$

(3) The identity element is 1.

(4) The inverse of $q \in O_5$, where $q = \frac{a}{b}$ can be defined as $q^{-1} = \frac{b}{a} \cdot 2^{-\lfloor \log_2(\frac{b}{a}) \rfloor}$.

(5) If the operation ab is commutative, then $a \cdot b \cdot 2^{-\lfloor \log_2(a \cdot b) \rfloor} = b \cdot a \cdot 2^{-\lfloor \log_2(b \cdot a) \rfloor}$. This is true because of the commutative property of multiplication itself: $a \cdot b = b \cdot a$. ■

Let $W_{2,3,5}$ be the set of ordered triples (a, b, c), where $a \in \mathbb{Z}$, $b \in \mathbb{Z}$, and $c \in \mathbb{Z}$, and where $a = -\lfloor \log_2(3^b \cdot 5^c) \rfloor$. Similarly, the group $W_{2,3}$ can be defined as the set of ordered pairs (a, b), where $a \in \mathbb{Z}$, and $b \in \mathbb{Z}$, and where $a = -\lfloor \log_2(3^b) \rfloor$. As $W_{2,3}$ behaves like the ordered triple $W_{2,3,5}$ where the third component is always 0, we need not provide separate proof of its behaviors.

Theorem 2.3.4 The set $W_{2,3,5}$ forms a commutative group under the operation

$$(a, b, c)(d, e, f) = (-\lfloor \log_2(3^{b+e} \cdot 5^{c+f}) \rfloor, b + e, c + f).$$

Proof 2.3.4

(1) If the group is closed, the operation $(a, b, c)(d, e, f)$ must yield another element of the set. First, the group product $(-\lfloor \log_2(3^{b+e} \cdot 5^{c+f}) \rfloor, b + e, c + f)$ must satisfy $-\lfloor \log_2(3^{b+e} \cdot 5^{c+f}) \rfloor \in \mathbb{Z}$, $b + e \in \mathbb{Z}$, and $c + f \in \mathbb{Z}$. Because $-\lfloor x \rfloor \in \mathbb{Z}$ for all x, and because $\mathbb{Z}$ is closed under addition, this condition is satisfied. In the definition of the set, the first component in the ordered triple is always defined in terms of the second and third components. It is clear that the group operation is designed to maintain this relationship.

(2) The group operation inherits associativity from ordinary addition of integers.

(3) The identity element is $(0, 0, 0)$.

(4) The inverse of $(a, b, c) \in W_{2,3,5}$ is $(-\lfloor \log_2(3^{-b} \cdot 5^{-c}) \rfloor, -b, -c)$.

(5) The group operation inherits commutativity from ordinary addition of integers. ■

Let $I_{12,7,3}$ be the set of ordered triples (a, b, c), where $a \in \mathbb{Z}_{12}$, $b \in \mathbb{Z}$, and $c \in \mathbb{Z}$. The group $\mathbb{Z}_{12}$ is familiar to music theorists as the mod-12 additive group of pitch-class set theory. Let us define the group $I_{12,7}$ in a similar fashion as the set of ordered pairs (a, b), where $a \in \mathbb{Z}_{12}$, and $b \in \mathbb{Z}$. As we observed with $W_{2,3,5}$ and $W_{2,3}$, $I_{12,7}$ behaves like the ordered triple $I_{12,7,3}$ where the third component is always 0.

Theorem 2.3.5 The set $I_{12,7,3}$ forms a commutative group under the operation

$$(a, b, c)(d, e, f) = ((a + d) \bmod 12, b + e, d + f).$$

Proof 2.3.5 As $\mathbb{Z}_{12}$ is a commutative group under addition modulo 12, and $\mathbb{Z}$ is a commutative group under addition, it follows that $I_{12,7,3}$ is also a commutative group. The identity element is $(0, 0, 0)$, and the inverse of $(a, b, c) \in I_{12,7,3}$ is $((11 \cdot a) \bmod 12, -b, -c)$. ■

While the first component of the groups $I_{12,7,3}$ and $I_{12,7}$ alone constitutes a finite set, the groups themselves do not have a finite number of elements. This is because there is an infinite number of possible diatonic spellings of any of the twelve pitch classes and, in 5-limit JI, an infinite number of syntonic-comma-shifted instances of each of these diatonic spellings. The groups O_5, $W_{2,3,5}$, and $I_{12,7,3}$ are thus isomorphic to the *Tonnetz*, or 5-limit tuning lattice, which will be discussed in Chapter 3. The maps between pitch space and pitch-class space can be defined as follows. First, o, which maps $N_5 \rightarrow O_5$ and $N_3 \rightarrow O_3$, is defined for all $q \in N_5$ or N_3 as $o(q) = q \cdot 2^{-\lfloor \log_2(q) \rfloor}$. In addition, the map $w : V_5 \rightarrow W_5$ is defined for all $(a, b, c) \in V_5$ as $w(a, b, c) = (-\lfloor \log_2(3^b \cdot 5^c) \rfloor, b, c)$. Finally, $i : H_{12,7,3} \rightarrow I_{12,7,3}$ is the map defined for $(d, e, f) \in H_{12,7,3}$ as $i(d, e, f) = (d \bmod 12, e - 7 \cdot \left\lfloor \frac{d}{12} \right\rfloor, f - 3 \cdot \left\lfloor \frac{d}{12} \right\rfloor)$.

Figure 2.9 shows the relationships among the 3-limit groups we have just defined and the syntonic-comma-restricted 5-limit groups from which the scale in Table 2.1 is drawn. As we have just seen, there is an infinite number of instances of any single spelling of a pitch in the 5-limit groups in Figure 2.8, all separated by the syntonic comma. In Figure 2.9, however, there is only one instance of any spelling of a pitch. Because 3-limit tuning also contains only a single instance of each spelling of a pitch, these 5-limit groups, D_5, $L_{2,3,5}$, $X_{12,7,3}$, E_5, $M_{2,3,5}$, and $Y_{12,7,3}$, are isomorphic to the already-defined 3-limit groups with which they are vertically aligned in the figure. (Note the double-headed vertical arrows in Figure 2.9.) Let D_5 be the set of positive rationals with prime limit 5 $q = 2^a \cdot 3^b \cdot 5^c$ satisfying the condition that $b - 2 \leq c \leq b + 2$. If q and $r \in D_5$, where $q = 2^a \cdot 3^b \cdot 5^c$ and $r = 2^d \cdot 3^e \cdot 5^f$, we can define

$$qr = 2^{a-4\cdot c+d-4\cdot f+4\cdot \left\lfloor \frac{b+4\cdot c+e+4\cdot f}{5} \right\rceil} \cdot 3^{b+4\cdot c+e+4\cdot f-4\cdot \left\lfloor \frac{b+4\cdot c+e+4\cdot f}{5} \right\rceil} \cdot 5^{\left\lfloor \frac{b+4\cdot c+e+4\cdot f}{5} \right\rceil}$$

as an operation on this set.[26]

Theorem 2.3.6 D_5 forms a commutative group. As such, it has the properties of (1) closure, (2) associativity, (3) identity, (4) inverse, and (5) commutativity.

Proof 2.3.6

(1) To show that the set is closed, we must confirm that, for q and $r \in D_5$, $qr = 2^a \cdot 3^b \cdot 5^c$ is a positive rational number with prime limit 5, and that $b - 2 \leq c \leq b + 2$. First, for all a, b, c, d, e, and $f \in \mathbb{Z}$,

$$2^{a-4\cdot c+d-4\cdot f+4\cdot \left\lfloor \frac{b+4\cdot c+e+4\cdot f}{5} \right\rceil} \cdot 3^{b+4\cdot c+e+4\cdot f-4\cdot \left\lfloor \frac{b+4\cdot c+e+4\cdot f}{5} \right\rceil} \cdot 5^{\left\lfloor \frac{b+4\cdot c+e+4\cdot f}{5} \right\rceil}$$

is, by definition, a positive rational number with prime limit 5. Also,

[26] See the entry on rounding in Appendix A for the rounding conventions to be used in this dissertation.

$$b+4\cdot c+e+4\cdot f-4\cdot\left\lfloor\frac{b+4\cdot c+e+4\cdot f}{5}\right\rceil-2 \;\leq\; \left\lfloor\frac{b+4\cdot c+e+4\cdot f}{5}\right\rceil \;\leq\; b+4\cdot c+e+4\cdot f-4\cdot\left\lfloor\frac{b+4\cdot c+e+4\cdot f}{5}\right\rceil+2,$$

because, if $m = b + 4 \cdot c + e + 4 \cdot f$, then

$$m - 4 \cdot \left\lfloor\frac{m}{5}\right\rceil - 2 \leq \left\lfloor\frac{m}{5}\right\rceil \leq m - 4 \cdot \left\lfloor\frac{m}{5}\right\rceil + 2.$$

An equivalent statement is

$$\left\lfloor\frac{m}{5}\right\rceil - 2 \leq m - 4\left\lfloor\frac{m}{5}\right\rceil \leq \left\lfloor\frac{m}{5}\right\rceil + 2.$$

We can assert that m has 5 distinct integral values for which $\left\lfloor\frac{m}{5}\right\rceil$ has a single integral value. More specifically, for a single value of $\left\lfloor\frac{m}{5}\right\rceil$, m can only be

$$\begin{aligned}
&(5\left\lfloor\frac{m}{5}\right\rceil - 2),\\
&(5\left\lfloor\frac{m}{5}\right\rceil - 1),\\
&(5\left\lfloor\frac{m}{5}\right\rceil),\\
&(5\left\lfloor\frac{m}{5}\right\rceil + 1), \text{or}\\
&(5\left\lfloor\frac{m}{5}\right\rceil + 2).
\end{aligned}$$

Substituting each of these values for m in the expression $m - 4\left\lfloor\frac{m}{5}\right\rceil$,

$$\begin{aligned}
(5\left\lfloor\frac{m}{5}\right\rceil - 2) - 4\left\lfloor\frac{m}{5}\right\rceil &= \left\lfloor\frac{m}{5}\right\rceil - 2\\
(5\left\lfloor\frac{m}{5}\right\rceil - 1) - 4\left\lfloor\frac{m}{5}\right\rceil &= \left\lfloor\frac{m}{5}\right\rceil - 1\\
(5\left\lfloor\frac{m}{5}\right\rceil \quad) - 4\left\lfloor\frac{m}{5}\right\rceil &= \left\lfloor\frac{m}{5}\right\rceil\\
(5\left\lfloor\frac{m}{5}\right\rceil + 1) - 4\left\lfloor\frac{m}{5}\right\rceil &= \left\lfloor\frac{m}{5}\right\rceil + 1\\
(5\left\lfloor\frac{m}{5}\right\rceil + 2) - 4\left\lfloor\frac{m}{5}\right\rceil &= \left\lfloor\frac{m}{5}\right\rceil + 2.
\end{aligned}$$

Therefore,

$$\left\lfloor\frac{m}{5}\right\rceil - 2 \leq m - 4\left\lfloor\frac{m}{5}\right\rceil \leq \left\lfloor\frac{m}{5}\right\rceil + 2,$$

as desired.

(2) The operation $qr = 2^{a-4\cdot c+d-4\cdot f+4\cdot\left\lfloor\frac{b+4\cdot c+e+4\cdot f}{5}\right\rceil} \cdot 3^{b+4\cdot c+e+4\cdot f-4\cdot\left\lfloor\frac{b+4\cdot c+e+4\cdot f}{5}\right\rceil} \cdot 5^{\left\lfloor\frac{b+4\cdot c+e+4\cdot f}{5}\right\rceil}$ is associative if, for all $q = 2^a \cdot 3^b \cdot 5^c$, $r = 2^d \cdot 3^e \cdot 5^f$, and $s = 2^g \cdot 3^h \cdot 5^i \in D_5$, we can show that $(qr)s = q(rs)$.

$$
\begin{aligned}
&(qr)s = \\
&2^{a-4c+d-4f+4\left\lfloor\frac{b+4c+e+4f}{5}\right\rceil-4\left\lfloor\frac{b+4c+e+4f}{5}\right\rceil+g-4i+4\left\lfloor\frac{b+4c+e+4f+h+4i}{5}\right\rceil} \cdot \\
&3^{b+4c+e+4f-4\left\lfloor\frac{b+4c+e+4f}{5}\right\rceil+4\left\lfloor\frac{b+4c+e+4f}{5}\right\rceil+h+4i-4\left\lfloor\frac{b+4c+e+4f+h+4i}{5}\right\rceil} \cdot \\
&5^{\left\lfloor\frac{b+4c+e+4f+h+4i}{5}\right\rceil} = \\
&2^{a-4c+d-4f+g-4i+4\left\lfloor\frac{b+4c+e+4f+h+4i}{5}\right\rceil} \cdot 3^{b+4c+e+4f+h+4i-4\left\lfloor\frac{b+4c+e+4f+h+4i}{5}\right\rceil} \cdot 5^{\left\lfloor\frac{b+4c+e+4f+h+4i}{5}\right\rceil}
\end{aligned}
$$

$$
\begin{aligned}
&q(rs) = \\
&2^{a-4c+d-4f+g-4i+4\left\lfloor\frac{e+4f+h+4i}{5}\right\rceil-4\left\lfloor\frac{e+4f+h+4i}{5}\right\rceil+4\left\lfloor\frac{b+4c+e+4f+h+4i}{5}\right\rceil} \cdot \\
&3^{b+4c+e+4f+h-4i-4\left\lfloor\frac{e+4f+h+4i}{5}\right\rceil+4\left\lfloor\frac{e+4f+h+4i}{5}\right\rceil-4\left\lfloor\frac{b+4c+e+4f+h+4i}{5}\right\rceil} \cdot \\
&5^{\left\lfloor\frac{b+4c+e+4f+h+4i}{5}\right\rceil} = \\
&2^{a-4c+d-4f+g-4i+4\left\lfloor\frac{b+4c+e+4f+h+4i}{5}\right\rceil} \cdot 3^{b+4c+e+4f+h+4i-4\left\lfloor\frac{b+4c+e+4f+h+4i}{5}\right\rceil} \cdot 5^{\left\lfloor\frac{b+4c+e+4f+h+4i}{5}\right\rceil}
\end{aligned}
$$

(3) The identity element is 1.

(4) The inverse of $q \in D_5$, where $q = \frac{a}{b}$ can be defined as $q^{-1} = \frac{b}{a}$.

(5) The operation qr inherits commutativity from conventional addition of integers. ■

Let E_5 be the set of positive rationals with prime limit 5 $q = 2^a \cdot 3^b \cdot 5^c$ where $1 \leq q < 2$ and where $b-2 \leq c \leq b+2$. For the set E_5 and for q and $r \in E_5$, where $q = 2^a \cdot 3^b \cdot 5^c$ and $r = 2^d \cdot 3^e \cdot 5^f$, we can define the operation

$$qr = 2^{-\left\lfloor\log_2\left(3^{b+4\cdot c+e+4\cdot f-4\cdot\left\lfloor\frac{b+4\cdot c+e+4\cdot f}{5}\right\rceil}\cdot 5^{\left\lfloor\frac{b+4\cdot c+e+4\cdot f}{5}\right\rceil}\right)\right\rfloor} \cdot 3^{b+4\cdot c+e+4\cdot f-4\cdot\left\lfloor\frac{b+4\cdot c+e+4\cdot f}{5}\right\rceil} \cdot 5^{\left\lfloor\frac{b+4\cdot c+e+4\cdot f}{5}\right\rceil}.$$

Theorem 2.3.7 E_5 forms a commutative group. As such, it has the properties of (1) closure, (2) associativity, (3) identity, (4) inverse, and (5) commutativity.

Proof 2.3.7

(1) The set is closed because of three properties. First, for all a, b, c, d, e, and $f \in \mathbb{Z}$,

$$2^{-\lfloor \log_2(3^{b+4\cdot c+e+4\cdot f-4\cdot\left\lfloor\frac{b+4\cdot c+e+4\cdot f}{5}\right\rceil}\cdot 5^{\left\lfloor\frac{b+4\cdot c+e+4\cdot f}{5}\right\rceil})\rfloor}\cdot 3^{b+4\cdot c+e+4\cdot f-4\cdot\left\lfloor\frac{b+4\cdot c+e+4\cdot f}{5}\right\rceil}\cdot 5^{\left\lfloor\frac{b+4\cdot c+e+4\cdot f}{5}\right\rceil}$$

is by definition a positive rational number with prime limit 5. Next, for q and $r \in D_5$ where $qr = 2^a \cdot 3^b \cdot 5^c$, proof 2.3.6 confirms in this case as well that $b-2 \leq c \leq b+2$. Finally, by definition, when

$$2^{-\lfloor \log_2(3^{b+4\cdot c+e+4\cdot f-4\cdot\left\lfloor\frac{b+4\cdot c+e+4\cdot f}{5}\right\rceil}\cdot 5^{\left\lfloor\frac{b+4\cdot c+e+4\cdot f}{5}\right\rceil})\rfloor},$$

is multiplied by any value of

$$3^{b+4\cdot c+e+4\cdot f-4\cdot\left\lfloor\frac{b+4\cdot c+e+4\cdot f}{5}\right\rceil}\cdot 5^{\left\lfloor\frac{b+4\cdot c+e+4\cdot f}{5}\right\rceil},$$

it gives a value qr such that $1 \leq qr < 2$.

(2) The operation qr is associative if, for all $q = 2^a \cdot 3^b \cdot 5^c$, $r = 2^d \cdot 3^e \cdot 5^f$, and $s = 2^g \cdot 3^h \cdot 5^i \in D_5$, we can show that $(qr)s = q(rs)$. Proofs 2.3.6 and 2.3.3 confirm this assertion.

(3) The identity element is 1.

(4) The inverse of $q \in D_5$, where $q = 2^a \cdot 3^b \cdot 5^c$ can be defined as $q^{-1} = 2^{-\lfloor \log_2(3^{-b}\cdot 5^{-c})\rfloor}\cdot 3^{-b}\cdot 5^{-c}$.

(5) The operation qr inherits commutativity from conventional addition of integers. ■

Let $L_{2,3,5}$ be the set of ordered triples (a, b, c) where a, b, and $c \in \mathbb{Z}$ and where $b-2 \leq c \leq b+2$.

Theorem 2.3.8 $L_{2,3,5}$ forms a commutative group under the operation

$$(a,b,c)(d,e,f)=\left(a-4\cdot c+d-4\cdot f+4\cdot\left\lfloor\frac{b+4\cdot c+e+4\cdot f}{5}\right\rceil,\ b+4\cdot c+e+4\cdot f-4\cdot\left\lfloor\frac{b+4\cdot c+e+4\cdot f}{5}\right\rceil,\ \left\lfloor\frac{b+4\cdot c+e+4\cdot f}{5}\right\rceil\right).$$

Proof 2.3.8

(1) To show that the set is closed, we must confirm that the product of two elements has two properties. First, $(a, b, c)(d, e, f)$ must also be in the set of ordered triples of integers.

This is clearly inherited from the closure property of conventional addition of integers. Second, we must show that, if $(a, b, c)(d, e, f) = (g, h, i)$, then $h - 2 \leq i \leq h + 2$. Proving that $i-2 \leq h \leq i+2$ would be equivalent. Let the integer $m = b+4c+e+4f$. Rewriting $i - 2 \leq h \leq i + 2$ in terms of m, we thus wish to establish that

$$\left\lfloor \frac{m}{5} \right\rceil - 2 \leq m - 4 \left\lfloor \frac{m}{5} \right\rceil \leq \left\lfloor \frac{m}{5} \right\rceil + 2.$$

We can assert that m has 5 distinct integral values for which $\left\lfloor \frac{m}{5} \right\rceil$ has a single integral value. More specifically, for a single value of $\left\lfloor \frac{m}{5} \right\rceil$, m can only be

$$\begin{aligned}
&(5 \left\lfloor \frac{m}{5} \right\rceil - 2),\\
&(5 \left\lfloor \frac{m}{5} \right\rceil - 1),\\
&(5 \left\lfloor \frac{m}{5} \right\rceil),\\
&(5 \left\lfloor \frac{m}{5} \right\rceil + 1), \text{or}\\
&(5 \left\lfloor \frac{m}{5} \right\rceil + 2).
\end{aligned}$$

Substituting each of these values for m in the expression $m - 4 \left\lfloor \frac{m}{5} \right\rceil$,

$$\begin{aligned}
(5 \left\lfloor \frac{m}{5} \right\rceil - 2) - 4 \left\lfloor \frac{m}{5} \right\rceil &= \left\lfloor \frac{m}{5} \right\rceil - 2\\
(5 \left\lfloor \frac{m}{5} \right\rceil - 1) - 4 \left\lfloor \frac{m}{5} \right\rceil &= \left\lfloor \frac{m}{5} \right\rceil - 1\\
(5 \left\lfloor \frac{m}{5} \right\rceil \quad) - 4 \left\lfloor \frac{m}{5} \right\rceil &= \left\lfloor \frac{m}{5} \right\rceil\\
(5 \left\lfloor \frac{m}{5} \right\rceil + 1) - 4 \left\lfloor \frac{m}{5} \right\rceil &= \left\lfloor \frac{m}{5} \right\rceil + 1\\
(5 \left\lfloor \frac{m}{5} \right\rceil + 2) - 4 \left\lfloor \frac{m}{5} \right\rceil &= \left\lfloor \frac{m}{5} \right\rceil + 2.
\end{aligned}$$

Therefore,

$$\left\lfloor \frac{m}{5} \right\rceil - 2 \leq m - 4 \left\lfloor \frac{m}{5} \right\rceil \leq \left\lfloor \frac{m}{5} \right\rceil + 2,$$

as desired.

(2) The set operation is associative if, for (a, b, c), (d, e, f), and $(g, h, i) \in L_{2,3,5}$, we can show that $((a, b, c)(d, e, f))(g, h, i) = (a, b, c)((d, e, f)(g, h, i))$.

$$
\begin{aligned}
&((a,b,c)(d,e,f))(g,h,i) = \\
&\Big(a - 4c + d - 4f + 4\left\lfloor \tfrac{b+4c+e+4f}{5} \right\rceil - 4\left\lfloor \tfrac{b+4c+e+4f}{5} \right\rceil + g - 4i + 4\left\lfloor \tfrac{b+4c+e+4f+h+4i}{5} \right\rceil, \\
&b + 4c + e + 4f - 4\left\lfloor \tfrac{b+4c+e+4f}{5} \right\rceil + 4\left\lfloor \tfrac{b+4c+e+4f}{5} \right\rceil + h + 4i - 4\left\lfloor \tfrac{b+4c+e+4f+h+4i}{5} \right\rceil, \\
&\left\lfloor \tfrac{b+4c+e+4f+h+4i}{5} \right\rceil \Big) = \\
&\Big(a - 4c + d - 4f + g - 4i + 4\left\lfloor \tfrac{b+4c+e+4f+h+4i}{5} \right\rceil, \\
&b + 4c + e + 4f + h + 4i - 4\left\lfloor \tfrac{b+4c+e+4f+h+4i}{5} \right\rceil, \\
&\left\lfloor \tfrac{b+4c+e+4f+h+4i}{5} \right\rceil \Big)
\end{aligned}
$$

$$
\begin{aligned}
&(a,b,c)((d,e,f)(g,h,i)) = \\
&\Big(a - 4c + d - 4f + g - 4i + 4\left\lfloor \tfrac{e+4f+h+4i}{5} \right\rceil - 4\left\lfloor \tfrac{e+4f+h+4i}{5} \right\rceil + 4\left\lfloor \tfrac{b+4c+e+4f+h+4i}{5} \right\rceil, \\
&b + 4c + e + 4f + h - 4i - 4\left\lfloor \tfrac{e+4f+h+4i}{5} \right\rceil + 4\left\lfloor \tfrac{e+4f+h+4i}{5} \right\rceil - 4\left\lfloor \tfrac{b+4c+e+4f+h+4i}{5} \right\rceil, \\
&\left\lfloor \tfrac{b+4c+e+4f+h+4i}{5} \right\rceil \Big) = \\
&\Big(a - 4c + d - 4f + g - 4i + 4\left\lfloor \tfrac{b+4c+e+4f+h+4i}{5} \right\rceil, \\
&b + 4c + e + 4f + h + 4i - 4\left\lfloor \tfrac{b+4c+e+4f+h+4i}{5} \right\rceil, \\
&\left\lfloor \tfrac{b+4c+e+4f+h+4i}{5} \right\rceil \Big)
\end{aligned}
$$

(3) The identity element is $(0,0,0)$.

(4) The inverse of $(a,b,c) \in L_{2,3,5}$ is $(-a,-b,-c)$.

(5) The operation $(a,b,c)(d,e,f)$ inherits commuativity from conventional addition of integers. ■

Let $M_{2,3,5}$ be the set of ordered triples (a,b,c) where a, b, and $c \in \mathbb{Z}$, where $a = -\lfloor \log_2(3^b \cdot 5^c) \rfloor$, and where $b - 2 \le c \le b + 2$.

Theorem 2.3.9 The set $M_{2,3,5}$ forms a commutative group under the operation

$$\begin{aligned}(a,b,c)(d,e,f) &= \Big(-\left\lfloor \log_2(3^{b+4\cdot c+e+4\cdot f-4\cdot\left\lfloor\frac{b+4\cdot c+e+4\cdot f}{5}\right\rceil}\cdot 5^{\left\lfloor\frac{b+4\cdot c+e+4\cdot f}{5}\right\rceil})\right\rfloor,\\ &\quad b+4\cdot c+e+4\cdot f-4\cdot\left\lfloor\frac{b+4\cdot c+e+4\cdot f}{5}\right\rceil,\\ &\quad \left\lfloor\frac{b+4\cdot c+e+4\cdot f}{5}\right\rceil\Big).\end{aligned}$$

Proof 2.3.9

(1) Proofs 2.3.4 and 2.3.8 can be reapplied here to show that this set is closed under the operation $(a,b,c)(d,e,f)$.

(2) Proofs 2.3.4 and 2.3.8 confirm that the operation is associative on the set.

(3) The identity element is $(0,0,0)$.

(4) The inverse of $(a,b,c)\in M_{2,3,5}$ is $(-\lfloor\log_2(3^{-b}\cdot 5^{-c})\rfloor,-b,-c)$.

(5) The set operation inherits commutativity from conventional addition of integers. ■

Let $X_{12,7,3}$ be the set of ordered triples (a,b,c) where a, b, and $c\in\mathbb{Z}$ and where $c=(3\cdot(-11\cdot a+19\cdot b)+5\cdot(7\cdot a-12\cdot b))-\left\lfloor\frac{7\cdot a-12\cdot b}{5}\right\rceil$.

Theorem 2.3.10 The set $X_{12,7,3}$ forms a commutative group under the operation

$$\begin{aligned}(a,b,c)(d,e,f) &= \Big(a+d,b+e,(3\cdot(-11\cdot(a+d)+19\cdot(b+e))+\\ &\quad 5\cdot(7\cdot(a+d)-12\cdot(b+e)))-\left\lfloor\frac{7\cdot(a+d)-12\cdot(b+e)}{5}\right\rceil\Big).\end{aligned}$$

Proof 2.3.10

(1) Closure is inherited from conventional addition of integers.

(2) Associativity is inherited from conventional addition of integers.

(3) The identity element is $(0,0,0)$.

(4) The inverse of $(a,b,c)\in M_{2,3,5}$ is

$$\Big(-a,-b,(3\cdot(-11\cdot(-a)+19\cdot(-b))+5\cdot(7\cdot(-a)-12\cdot(-b)))-\left\lfloor\frac{7\cdot(-a)-12\cdot(-b)}{5}\right\rceil\Big).$$

(5) The set operation inherits commutativity from conventional addition of integers. ■

$Y_{12,7,3}$ can be defined as the set of ordered triples (a, b, c) where $a \in \mathbb{Z}_{12}$, where b and $c \in \mathbb{Z}$, and where $c = (3 \cdot (-11 \cdot a + 19 \cdot b) + 5 \cdot (7 \cdot a - 12 \cdot b)) - \left\lfloor \frac{7 \cdot a - 12 \cdot b}{5} \right\rceil$.

Theorem 2.3.11 The set $Y_{12,7,3}$ forms a commutative group under the operation

$$\begin{aligned}(a, b, c)(d, e, f) &= \Big((a + d) \bmod 12, b + e, (3 \cdot (-11 \cdot (a + d) + 19 \cdot (b + e)) + \\ &\quad 5 \cdot (7 \cdot (a + d) - 12 \cdot (b + e))) - \left\lfloor \frac{7 \cdot (a + d) - 12 \cdot (b + e)}{5} \right\rceil \Big).\end{aligned}$$

Proof 2.3.11

(1) Closure is inherited from conventional addition of integers and addition mod 12 over $\mathbb{Z}_{12}$.

(2) Associativity is inherited from conventional addition of integers and addition mod 12.

(3) The identity element is $(0, 0, 0)$.

(4) The inverse of $(a, b, c) \in M_{2,3,5}$ is

$$\Big(12 - a, -b, (3(-11(12 - a) + 19(-b)) + 5(7(12 - a) - 12(-b))) - \left\lfloor \frac{7(12 - a) - 12(-b)}{5} \right\rceil \Big).$$

(5) The set operation inherits commutativity from conventional addition of integers and addition mod 12. ■

The map $\delta : N_3 \to D_5$ and $O_3 \to E_5$ is defined for $q \in N_3$ or $q \in O_3$, where $q = 2^a \cdot 3^b$, as $\delta(q) = q \cdot (\frac{80}{81})^{\lfloor \frac{b}{5} \rceil}$. The map $\mu : V_{2,3} \to L_{2,3,5}$ and $W_{2,3} \to M_{2,3,5}$ is defined for $(a, b) \in V_{2,3}$ or $(a, b) \in W_{2,3}$ as $(a + 4 \cdot \lfloor \frac{b}{5} \rceil, b - 4 \cdot \lfloor \frac{b}{5} \rceil, \lfloor \frac{b}{5} \rceil)$. Lastly, the map $y : H_{12,7} \to X_{12,7,3}$ and $I_{12,7} \to Y_{12,7,3}$ is defined for $(d, e) \in H_{12,7}$ or $(d, e) \in I_{12,7}$ as

$$y(d, e) = \left(d, e, (3 \cdot (-11 \cdot d + 19 \cdot e) + 5 \cdot (7 \cdot d - 12 \cdot e)) - \left\lfloor \frac{7 \cdot d - 12 \cdot e}{5} \right\rceil \right).$$

Since this map takes any interval or pitch and its diatonic spelling and returns the corresponding value in the set from which scale-based JI derives its scale, the next section will discuss the mathematical derivation of the ratios in scale-based JI.

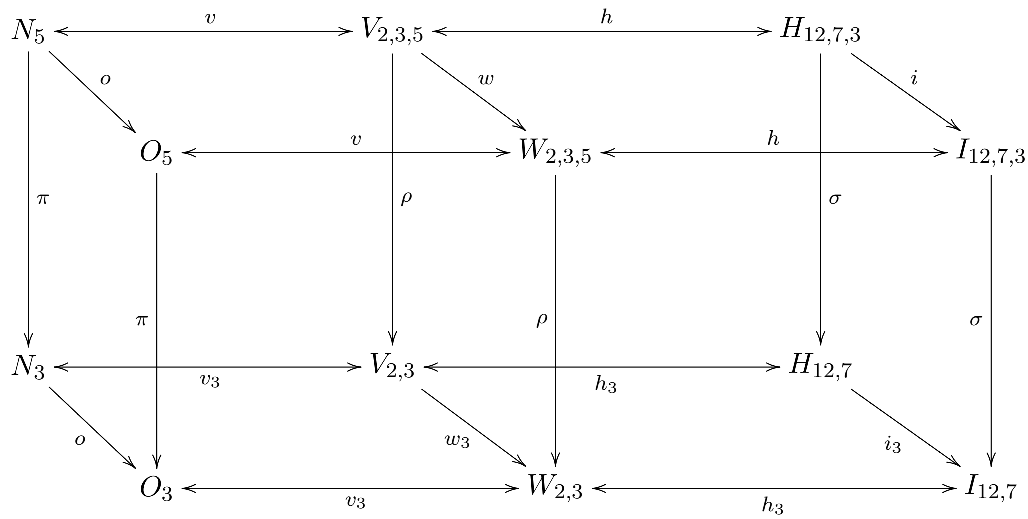


Figure 2.8: Homomorphisms among 5-Limit and 3-Limit JI and 12-, 7-, and 3-Tone Scales

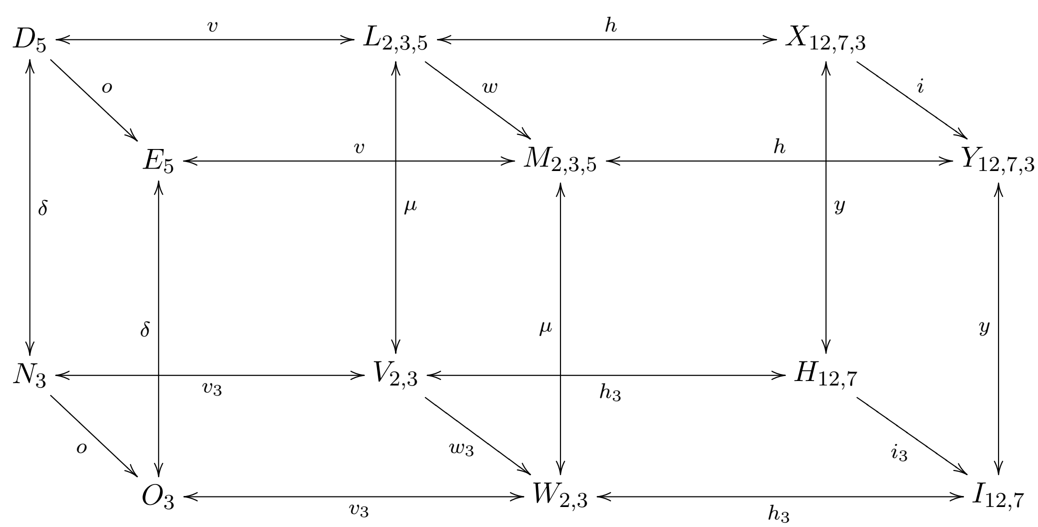


Figure 2.9: Isomorphism between the 12-Tone/7-Tone System and the 5-Limit Scale

2.4 Generalizing Diatonic Structures into 5-Limit JI

The previous section provided a mathematical definition of the groups of JI and diatonic spelling. Further, we examined a restricted 5-limit system isomorphic to Pythagorean tuning (3-limit JI). This 5-limit system offers purely tuned triads without creating an infinite number of pitch replications distinguished by the syntonic comma.[27] The repertoire of scale degrees given in Table 2.1 for use in scale-based JI is drawn from the octave-reduced instance of this system (E_5). In the following section we shall formally define this 12-member subset of O_5 (octave-reduced 5-limit JI) and E_5 (octave-reduced syntonic-comma-restricted 5-limit JI) that was first given in Table 2.1. The formalization of the 5-limit scale here offers a mathematical procedure for spelling and tuning any tonal piece of music given only the 12-tone pitch or pitch-class integers, the key of the piece, and the tonicizations and modulations involved. Figure 2.10 displays the isomorphic groups involved and the maps between them. The set of integers $\mathbb{Z}$ forms a group under the operation of addition. Its identity element is 0, and the inverse of $a \in \mathbb{Z}$ is $-a$. We shall use this set to measure ordered pitch intervals with 12 steps per octave. The octave-reduced set $\mathbb{Z}_{12}$ forms a group under the operation of addition modulo 12. Its identity element is 0, and the inverse of $a \in \mathbb{Z}_{12}$ may be derived from either $12 - a$ or $(11 \cdot a) \bmod 12$.

As we discovered in the previous section, the diatonic spelling of pitches or intervals is adequate for representing 3-limit JI (Pythagorean tuning). The sets $B_{12,7}$, $P_{2,3}$, F_3, $C_{12,7}$, $Q_{2,3}$, and G_3 thus all form a Pythagorean system based on the scale-degree spellings in Table 2.1. As we shall see, the octave-reduced groups $C_{12,7}$, $Q_{2,3}$ and G_3 follow quite naturally from $B_{12,7}$, $P_{2,3}$ and F_3. The sets $B_{12,7}$ and $C_{12,7}$ represent the spelling of the intervals represented by $\mathbb{Z}$. First, $B_{12,7}$ is defined as the set of ordered pairs (a, b) where both a and $b \in \mathbb{Z}$ and b is always $\frac{7 \cdot a - (((7 \cdot a + 5) \bmod 12) - 5)}{12}$. Similarly, $C_{12,7}$ is defined as the set of ordered pairs (a, b) where $a \in \mathbb{Z}_{12}$ and $b \in \mathbb{Z}_7$ and b is always $\frac{7 \cdot a - (((7 \cdot a + 5) \bmod 12) - 5)}{12}$.

Theorem 2.4.1 The set $B_{12,7}$ forms a commutative group under the operation

$$(a, b)(c, d) = \left(a + c, \frac{7 \cdot (a + c) - (((7 \cdot (a + c) + 5) \bmod 12) - 5)}{12}\right),$$

and the set $C_{12,7}$ forms a commutative group under the operation

$$(a, b)(c, d) = \Big(a + c, ((((7 \cdot (a + c \bmod 12) + 5) \bmod 12) - 5) \cdot 4) \bmod 7\Big).$$

[27] Because not every triad is just in this system, we must make use of pitches outside the system in the manner that we discussed in Section 2.1.

Proof 2.4.1

(1) Closure is inherited from conventional addition of integers, addition mod 12 over $\mathbb{Z}_{12}$, and addition mod 7 over $\mathbb{Z}_7$.

(2) Associativity is inherited from conventional addition of integers, addition mod 12, and addition mod 7.

(3) The identity element of both sets is $(0,0)$.

(4) The inverse of $(a,b) \in B_{12,7}$ is $(-a, \frac{-7\cdot a-(((-7\cdot a+5) \bmod 12)-5)}{12}$, and the inverse of $(a,b) \in C_{12,7}$ is $(12-a, ((((7\cdot((12-a) \bmod 12)+5) \bmod 12)-5)\cdot 4) \bmod 7)$.

(5) The set operations inherit commutativity from conventional addition of integers, addition mod 12, and addition mod 7. ■

Comparison of Figure 2.10 with Figure 2.8 and Figure 2.9 suggests that $P_{2,3}$ and $Q_{2,3}$ use ordered pairs to represent the powers of 2 and 3 in the 3-limit frequency ratios that correspond with the spellings in Table 2.1. Let $P_{2,3}$ be the set of ordered pairs (a,b) where $a \in \mathbb{Z}$ and b is a member of the set of integers mod 12 -5; that is, b is always an integer such that $-5 \leq b < 7$. The set $Q_{2,3}$ is defined as the set of ordered pairs (a,b) where $a \in \mathbb{Z}$, b is a member of the set of integers mod 12 -5, and a is always $-\lfloor \log_2(3^b) \rfloor$.

Theorem 2.4.2 The set $P_{2,3}$ forms a commutative group under the operation

$$(a,b)(c,d) = \Big(a+c,\ ((b+d+5) \bmod 12)-5\Big),$$

and the set $Q_{2,3}$ forms a commutative group under the operation

$$(a,b)(c,d) = \Big(-\lfloor \log_2(3^{((b+d+5) \bmod 12)-5}) \rfloor,\ ((b+d+5) \bmod 12)-5\Big).$$

Proof 2.4.2

(1) Closure is inherited from conventional addition of integers, from addition mod 12, and, in the case of $Q_{2,3}$, from $W_{2,3}$.

(2) Associativity is inherited from conventional addition of integers, from addition mod 12, and, in the case of $Q_{2,3}$, from $W_{2,3}$.

(3) The identity element of both sets is $(0,0)$.

(4) The inverse of $(a, b) \in P_{2,3}$ is $(-a, ((17-b) \bmod 12) - 5)$; and the inverse of $(a, b) \in Q_{2,3}$ is $(-\lfloor \log_2(3^{((17-b) \bmod 12)-5}) \rfloor, ((17-b) \bmod 12) - 5)$.

(5) The set operations inherit commutativity from conventional addition of integers, from addition mod 12, and, in the case of $Q_{2,3}$, from $W_{2,3}$. ■

The 3-limit frequency ratios represented by the row vectors in $P_{2,3}$ form the elements of F_3. We shall define F_3 as the set of positive rational numbers q with prime limit 3 where $q = 2^a \cdot 3^b$, where $a \in \mathbb{Z}$, and where b is a member of the set of integers mod 12 -5. This set clearly has the same relationship to $P_{2,3}$ as N_3 has to $V_{2,3}$. Likewise, G_3 is defined as the set of positive rationals with prime limit 3 $q = 2^a \cdot 3^b$ such that a is always $a = -\lfloor \log_2(3^b) \rfloor$.

Theorem 2.4.3 Given $q = 2^a \cdot 3^b$, and $r = 2^d \cdot 3^e$ (both $\in F_3$), the set F_3 forms a commutative group under the operation

$$qr = 2^{a+d} \cdot 3^{((b+e+5) \bmod 12)-5}.$$

Given q and $r \in G_3$, the set G_3 also forms a commutative group under the operation

$$qr = 2^{-\lfloor \log_2(3^{((b+e+5) \bmod 12)-5}) \rfloor} \cdot 3^{((b+e+5) \bmod 12)-5}.$$

Proof 2.4.3

(1) Several conditions must obtain for us to be able to assert closure. First, the result of the operation qr must be a positive 3-limit rational number. Clearly this is the case for both F_3 and G_3. We must also be able to confirm that the result $s = 2^g \cdot 3^h$ of the operation qr satisfies the condition $h \in \{\mathbb{Z}_{12} - 5\}$. In the operations on both F_3 and G_3 this is clearly true because $h = ((b+e+5) \bmod 12) - 5$ in both operations qr. Further, in the operation $qr = s$ on the set F_3, g must be an integer. This is true because of the closure property of addition over $\mathbb{Z}$. Finally, in the operation $qr = s$ on the set G_3, g must be $-\lfloor \log_2(3^h) \rfloor$. This is clearly true, as $g = -\lfloor \log_2(3^{((b+e+5) \bmod 12)-5}) \rfloor$.

(2) Associativity is inherited from from addition mod 12, conventional addition of integers in the case of F_3, and, in the case of G_3, from O_3.

(3) The identity element of both sets is 1.

(4) The inverse of $q = 2^a \cdot 3^b \in F_3$ is $2^{-a} \cdot 3^{((17-b) \bmod 12)-5}$; and the inverse of $q = 2^a \cdot 3^b \in G_3$ is $2^{-\lfloor \log_2(3^{((17-b) \bmod 12)-5}) \rfloor} \cdot 3^{((17-b) \bmod 12)-5}$.

(5) The set operations inherit commuatativity from addition mod 12, conventional addition of integers in the case of F_3, and, in the case of G_3, from O_3. ■

Of course the ratios that accompany the diatonic spellings in Table 2.1 are not 3-limit ratios, but rather 5-limit ratios. The groups $J_{12,7,3}$, $K_{12,7,3}$, $Z_{2,3,5}$, $A_{2,3,5}$, R_5, and S_5 thus all encode (in different ways) the structural relationships of the set of 5-limit ratios found in Table 2.1. $J_{12,7,3}$ is defined as the set of ordered triples (a, b, c) such that a, b, and $c \in \mathbb{Z}$, b is always

$$\frac{7 \cdot a - (((7 \cdot a + 5) \bmod 12) - 5)}{12},$$

and c is always

$$(3 \cdot (-11 \cdot a + 19 \cdot b) + 5 \cdot (7 \cdot a - 12 \cdot b)) - \left\lfloor \frac{7 \cdot a - 12 \cdot b}{5} \right\rceil.$$

$K_{12,7,3}$ is defined as the set of ordered triples (a, b, c) where $a \in \mathbb{Z}_{12}$, $b \in \mathbb{Z}_7$, and $c \in \mathbb{Z}$. Further, b is always

$$\frac{7 \cdot a - (((7 \cdot a + 5) \bmod 12) - 5)}{12},$$

and c is always

$$(3 \cdot (-11 \cdot a + 19 \cdot b) + 5 \cdot (7 \cdot a - 12 \cdot b)) - \left\lfloor \frac{7 \cdot a - 12 \cdot b}{5} \right\rceil.$$

Theorem 2.4.4 Given

$$j = \frac{7 \cdot (a + d) - (((7 \cdot (a + d) + 5) \bmod 12) - 5)}{12}$$

and

$$k = 3 \cdot (-11 \cdot (a + d) + 19 \cdot j) + 5 \cdot (7 \cdot (a + d) - 12 \cdot j) - \left\lfloor \frac{7 \cdot (a + d) - 12 \cdot j}{5} \right\rceil,$$

the set $J_{12,7,3}$ forms a commutative group under the operation

$$(a, b, c)(d, e, f) = (a + d, j, k).$$

Given

$$j = \frac{7 \cdot ((a + d) \bmod 12) - (((7 \cdot (a + d) + 5) \bmod 12) - 5)}{12}$$

and

$$k = 3(-11((a+d) \bmod 12)+19j)+5(7((a+d) \bmod 12)-12j)-\left\lfloor\frac{7((a+d) \bmod 12)-12j}{5}\right\rceil,$$

the set $K_{12,7,3}$ forms a commutative group under the operation

$$(a,b,c)(d,e,f) = ((a+d) \bmod 12, j, k).$$

Proof 2.4.4

(1) Studying the group operation confirms that it is designed to maintain the set's closure.

(2) Associativity is inherited from conventional addition of integers, and, in the case of $K_{12,7,3}$, addition mod 12 over $\mathbb{Z}_{12}$ and addition mod 7 over $\mathbb{Z}_7$.

(3) The identity element of both sets is $(0,0,0)$.

(4) Given

$$j^{-1} = \frac{-7\cdot a - (((-7\cdot a+5) \bmod 12)-5)}{12}$$

and

$$k^{-1} = (3\cdot(-11\cdot(-a)+19\cdot j^{-1})+5\cdot(7\cdot(-a)-12\cdot j^{-1})) - \left\lfloor\frac{7\cdot(-a)-12\cdot j^{-1}}{5}\right\rceil,$$

the inverse of $(a,b,c)\in J_{12,7,3}$ is $(-a, j^{-1}, k^{-1})$; and, given

$$j^{-1} = \frac{7\cdot(12-a) - (((7\cdot(12-a)+5) \bmod 12)-5)}{12}$$

and

$$k^{-1} = 3\cdot(-11(12-a)+19j^{-1})+5\cdot(7(12-a)-12j^{-1}) - \left\lfloor\frac{7(12-a)-12j^{-1}}{5}\right\rceil,$$

the inverse of $(a,b,c)\in K_{12,7,3}$ is $(12-a, j^{-1}, k^{-1})$.

(5) Commutativity is inherited from conventional addition of integers, and, in the case of $K_{12,7,3}$, addition mod 12 over $\mathbb{Z}_{12}$ and addition mod 7 over $\mathbb{Z}_7$. ■

$Z_{2,3,5}$ is defined as the set of ordered triples (a, b, c) such that $a \in \mathbb{Z}$, b is a member of the set of integers mod 12 -5, and $c \in \{-1, 0, 1\}$. $A_{2,3,5}$ is defined as the set of ordered triples (a, b, c) where $a \in \mathbb{Z}$, b is a member of the set of integers mod 12 -5, and $c \in \{-1, 0, 1\}$. Further, a is always $-\lfloor \log_2(3^b \cdot 5^c) \rfloor$.

Theorem 2.4.5 Given

$$j = \left\lfloor \frac{((b + 4 \cdot c + e + 4 \cdot f + 5) \bmod 12) - 5}{5} \right\rceil$$

and

$$k = ((b + 4 \cdot c + e + 4 \cdot f + 5) \bmod 12) - 5 - 4 \cdot j,$$

the set $Z_{2,3,5}$ forms a commutative group under the operation

$$(a, b, c)(d, e, f) = (a - 4 \cdot c + d - 4 \cdot f + 4 \cdot j, k, j).$$

Given

$$j = \left\lfloor \frac{((b + 4 \cdot c + e + 4 \cdot f + 5) \bmod 12) - 5}{5} \right\rceil$$

and

$$k = ((b + 4 \cdot c + e + 4 \cdot f + 5) \bmod 12) - 5 - 4 \cdot j,$$

the set $A_{2,3,5}$ forms a commutative group under the operation

$$(a, b, c)(d, e, f) = (-\lfloor \log_2(3^k \cdot 5^j) \rfloor, k, j).$$

Proof 2.4.5

(1) Closure is inherited from conventional addition of integers, addition mod 12 over $\mathbb{Z}_{12}$, and, in the case of $A_{2,3,5}$, $W_{2,3,5}$.

(2) Associativity is inherited from conventional addition of integers, addition mod 12 over $\mathbb{Z}_{12}$, and, in the case of $A_{2,3,5}$, $W_{2,3,5}$.

(3) The identity element of both sets is $(0, 0, 0)$.

(4) Given

$$j^{-1} = \left\lfloor \frac{((-b - 4 \cdot c + 5) \bmod 12) - 5}{5} \right\rceil$$

and

$$k^{-1} = ((-b - 4 \cdot c + 5) \bmod 12) - 5 - 4 \cdot j^{-1},$$

the inverse of $(a, b, c) \in Z_{2,3,5}$ is

$$(-a + 4 \cdot c + 4 \cdot j^{-1}, k^{-1}, j^{-1});$$

and, given

$$j^{-1} = \left\lfloor \frac{((-b - 4 \cdot c + 5) \bmod 12) - 5}{5} \right\rceil$$

and

$$k^{-1} = ((-b - 4 \cdot c + 5) \bmod 12) - 5 - 4 \cdot j^{-1},$$

the inverse of $(a, b, c) \in A_{2,3,5}$ is

$$(-\lfloor \log_2(3^{k^{-1}} \cdot 5^{j^{-1}}) \rfloor, k^{-1}, j^{-1}).$$

(5) Commutativity is inherited from conventional addition of integers, addition mod 12 over $\mathbb{Z}_{12}$, and, in the case of $A_{2,3,5}$, $W_{2,3,5}$.■

R_5 is defined as the set of positive rationals with prime limit 5 $q = 2^a \cdot 3^b \cdot 5^c$ such that $a \in \mathbb{Z}$, b is a member of the set of integers mod 12 -5, and $c \in \{-1, 0, 1\}$. S_5 is defined as the set of positive rationals with prime limit 5 $q = 2^a \cdot 3^b \cdot 5^c$ where $a \in \mathbb{Z}$, b is a member of the set of integers mod 12 -5, and $c \in \{-1, 0, 1\}$. Further, a is always $-\lfloor \log_2(3^b \cdot 5^c) \rfloor$.

Theorem 2.4.6 Given q and $r \in R_5$, such that $q = 2^a \cdot 3^b \cdot 5^c$ and $r = 2^d \cdot 3^e \cdot 5^f$, and further given

$$j = \left\lfloor \frac{((b + 4 \cdot c + e + 4 \cdot f + 5) \bmod 12) - 5}{5} \right\rceil$$

and

$$k = ((b + 4 \cdot c + e + 4 \cdot f + 5) \bmod 12) - 5 - 4 \cdot j,$$

the set R_5 forms a commutative group under the operation

$$qr = 2^{a - 4 \cdot c + d - 4 \cdot f + 4 \cdot j} \cdot 3^k \cdot 5^j.$$

Given q and $r \in S_5$, such that $q = 2^a \cdot 3^b \cdot 5^c$ and $r = 2^d \cdot 3^e \cdot 5^f$, and further given

$$j = \left\lfloor \frac{((b + 4 \cdot c + e + 4 \cdot f + 5) \bmod 12) - 5}{5} \right\rceil$$

and

$$k = ((b + 4 \cdot c + e + 4 \cdot f + 5) \bmod 12) - 5 - 4 \cdot j,$$

the set S_5 forms a commutative group under the operation

$$qr = 2^{-\lfloor \log_2(3^k \cdot 5^j) \rfloor} \cdot 3^k \cdot 5^j.$$

Proof 2.4.6

(1) Closure is inherited from addition of integers, addition mod 12 over $\mathbb{Z}_{12}$, N_5, and O_5.

(2) Associativity is also inherited from addition of integers, addition mod 12 over $\mathbb{Z}_{12}$, N_5, and O_5.

(3) The identity element of both sets is 1.

(4) Given

$$j^{-1} = \left\lfloor \frac{((-b - 4 \cdot c + 5) \bmod 12) - 5}{5} \right\rceil$$

and

$$k^{-1} = ((-b - 4 \cdot c + 5) \bmod 12) - 5 - 4 \cdot j^{-1},$$

the inverse of $q \in R_5$ is

$$2^{-a+4 \cdot c+4 \cdot j^{-1}} \cdot 3^{k^{-1}} \cdot 5^{j^{-1}};$$

and, given

$$j^{-1} = \left\lfloor \frac{((-b - 4 \cdot c + 5) \bmod 12) - 5}{5} \right\rceil$$

and

$$k^{-1} = ((-b - 4 \cdot c + 5) \bmod 12) - 5 - 4 \cdot j^{-1},$$

the inverse of $q \in S_5$ is

$$2^{-\lfloor \log_2(3^{k^{-1}} \cdot 5^{j^{-1}}) \rfloor} \cdot 3^{k^{-1}} \cdot 5^{j^{-1}}.$$

(5) Commutativity is inherited from addition of integers, addition mod 12 over $\mathbb{Z}_{12}$, N_5, and O_5.■

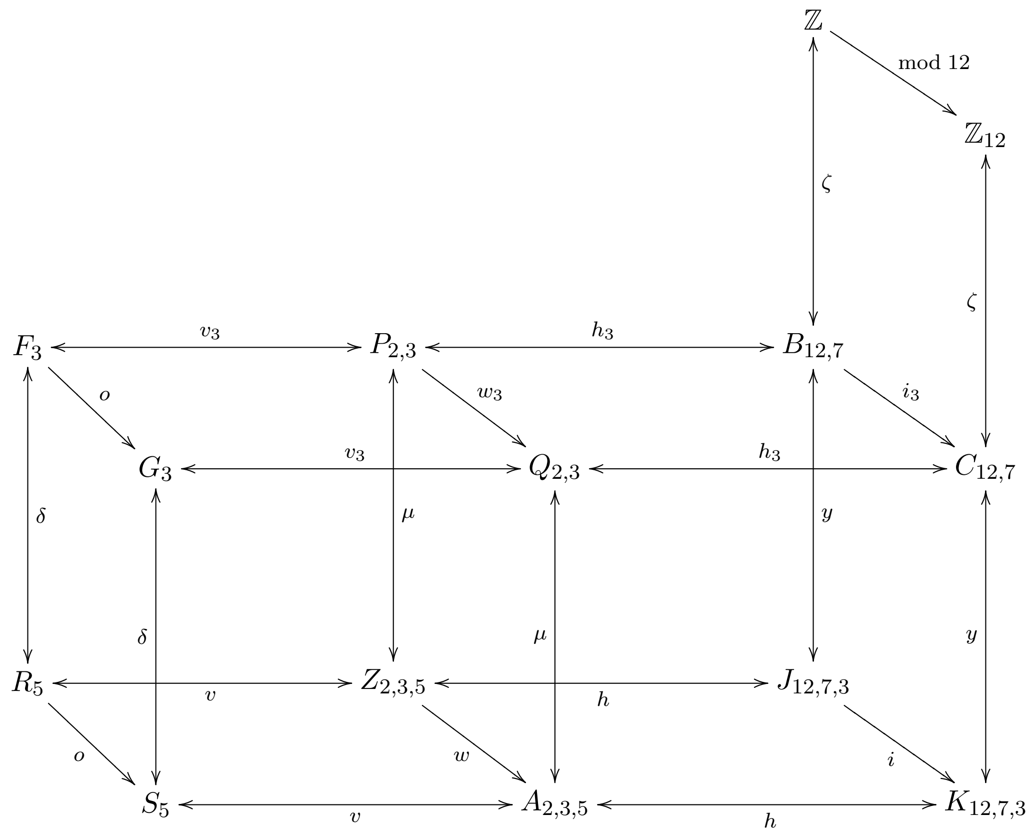


Figure 2.10: Isomorphism between the 12-Tone Scale and Scale-Based JI

Given the key of the music and the 12-tone pitch or pitch-class numbers of every note in a composition, one can correctly spell and tune the entire piece in scale-based JI using the maps ζ and y. The function y has already been defined in the previous section. Given an integer $a \in \mathbb{Z}$ or $a \in \mathbb{Z}_{12}$, the map ζ can be defined as

$$\zeta(a) = \left(a, \frac{7 \cdot a - (((7 \cdot a + 5) \bmod 12) - 5)}{12}\right).$$

Thus, given an interval measured from the tonic pitch, expressed as $a \in \mathbb{Z}$, the function $y(\zeta(a))$ gives the 12-, 7-, and 3-tone values necessary for correctly rendering the interval in scale-based 5-limit JI. Because this method assumes the identity element (0,0,0) to be the tonic pitch, some modifications must be made to these functions if one wishes to perform this transformation when the tonic is a value other than the identity element. This situation would arise, for example, in a piece that modulates. The passage in the subsidiary key area would require measurement from a non-identity-element tonic. For this purpose, it will be useful to make use of the fixed-zero convention from pitch-class set theory—that is, the identity element $(0,0,0) \in K_{12,7,3}$ is always the pitch class C. Moreover, we shall (arbitrarily) define the identity element $(0,0,0) \in J_{12,7,3}$ as the pitch C4 (middle C). The ordered triple $t = (t_1, t_2, t_3)$ will refer to the ordered triple of the current tonic pitch (class) as expressed as an element of $K_{12,7,3}$ or $J_{12,7,3}$. The new functions to work with this fixed-C representation of $H_{12,7,3}$ and $I_{12,7,3}$ are

$$\zeta_t(a) = \left(a,\ \frac{7 \cdot a - ((((7 \cdot ((a - t_1) \bmod 12) + 5) \bmod 12) - 5) + 7 \cdot t_1 - 12 \cdot t_2)}{12}\right)$$

and

$$y_t(a,b) = \left(a,\ b,\ (3\cdot(-11\cdot(a-t_1)+19\cdot(b-t_2))+5\cdot(7\cdot(a-t_1)-12\cdot(b-t_2))) - \left\lfloor \frac{7\cdot(a-t_1)-12\cdot(b-t_2)}{5} \right\rceil + t_3\right).$$

In order to arrive at an ordered triple in $H_{12,7,3}$ for any note in any key, one must begin by using the defined value for (middle) C as the first t to "bootstrap" oneself to the correct ordered triples for the keys and pitches.[28] After a brief explanation of how the function ζ_t transforms 12-tone pitch integers into their diatonic spellings and how y_t then gives the 3-tone component, we shall then explore the use of this procedure with two concrete examples.

The function ζ_t is simply a mathematical expression of Rule 4 from Table 2.2. First, ζ_t translates the pitch or pitch-class number into an expression of the number of fifths away from tonic by multiplying by 7 mod 12, and converting 7 through 11 fifths above tonic to -5 through -1 fifths below tonic. Then, ζ_t converts the number of fifths away from tonic into step numbers by subtracting the result from 7 times the original pitch number and dividing the result by 12. In other words, the key-defined step number for any pitch number is equal to 7 times the pitch number, minus the distance in fifths away from tonic, all divided by 12. The function y_t returns the tuning of any pitch in the context of a tonic by finding its number of fifths away from tonic and dividing its frequency by 1 syntonic comma for every 5 fifths away from tonic (starting at 3 fifths away). Note that y correctly generates the 3-tone component for any pitch with a specified diatonic spelling, regardless of whether it is in the

[28]Because the function measures distance in tonal space, the inital key in any analysis will thus be as closely related to C major in 5-limit JI as possible. The tuning lattice, which we shall examine in Chapter 3, allows these relationships to be visualized.

12-note scale given by ζ. This is important because ζ_t can return values outside of that 12-note scale. Moreover, the functions ζ_t and y_t cannot completely replace the procedure given in Table 2.2. One must still use Rules 2–4 to decide which scale's tonic pitch to use in the calculations. Further, to make the comma corrections that render the problematic chords in the system pure, the changes of scale prescribed by Rule 4 in Table 2.2 for non-modulatory passages must also be included in the calculations in the same manner as modulations and tonicizations.

The coordinates that result from applying the functions ζ_t and y_t will be used in the next chapter to graph musical progressions on a Cartesian plane and in transformation networks. Two examples of the use of ζ_t and y_t will therefore help to clarify how scale-based JI can be calculated mathematically. First, we shall examine the mathematical derivation of the spelling and tuning of the simple diatonic progression D: I vi ii V I. Because the pitch classes in the progression must be judged from a tonic pitch class other than C, our first step is to use the functions $\zeta_{(0,0)}$ and $y_{(0,0,0)}$ to obtain a t value for the key of D. Applying the pitch class integer of D, 2, to the function $\zeta_{(0,0)}$ gives the expression $\zeta_{(0,0)}(2) = (2, 1)$. Then applying the result (2,1) to the function $y_{(0,0,0)}$ generates $y_{(0,0,0)}(2, 1) = (2, 1, 1)$. The value of t we shall rely upon for much of this progression will thus be $t = (2, 1, 1)$. Next, we shall use the functions $\zeta_{(2,1)}$ and $y_{(2,1,1)}$ to obtain the $I_{12,7,3}$ value of each member of the first chord. The members of the tonic chord in D major, pitch classes 2, 6, and 9, generate the expressions $y_{(2,1,1)}(\zeta_{(2,1)}(2)) = (2, 1, 1)$, $y_{(2,1,1)}(\zeta_{(2,1)}(6)) = (6, 3, 2)$, and $y_{(2,1,1)}(\zeta_{(2,1)}(9)) = (9, 5, 3)$, respectively. In order to calculate values for the vi chord, we must consult Rule 4 of Table 2.2 to determine if a new t is needed. Since the vi harmony is a non-dominant chord in the original key, Rule 4 dictates that we use the closest chord member (other than the seventh) to $\hat{1}$ and $\hat{5}$ on the line of fifths as the new t value. This happens to be $\hat{1}$ in this case, so no change of t is required. The expressions for pitch classes 11, 2, and 6 are thus $y_{(2,1,1)}(\zeta_{(2,1)}(11)) = (11, 6, 3)$, $y_{(2,1,1)}(\zeta_{(2,1)}(2)) = (2, 1, 1)$, and $y_{(2,1,1)}(\zeta_{(2,1)}(6)) = (6, 3, 2)$. Following the same procedure for the ii chord, we find that we must use $\hat{4}$ as the new t value, since it is the closest chord member to $\hat{1}$ on the line of fifths. Therefore, we calculate the new t in terms of the old t by deriving $y_{(2,1,1)}(\zeta_{(2,1)}(7)) = (7, 4, 2)$. Using the new t for the ii chord, we obtain $y_{(7,4,2)}(\zeta_{(7,4)}(4)) = (4, 2, 1)$, $y_{(7,4,2)}(\zeta_{(7,4)}(7)) = (7, 4, 2)$, and $y_{(7,4,2)}(\zeta_{(7,4)}(11)) = (11, 6, 3)$. The change of t to $(7, 4, 2)$ in this case gives a different $\hat{2}$ $(4, 2, 1)$ from the $\hat{2}$ generated by $y_{(2,1,1)}(\zeta_{(2,1)}(4)) = (4, 2, 2)$. Because the next chord is a dominant chord, Rule 2 of Table 2.2 requires that its t value be taken from the key in which it is functioning $(2, 1, 1)$. Thus the V chord's values are $y_{(2,1,1)}(\zeta_{(2,1)}(9)) = (9, 5, 3)$, $y_{(2,1,1)}(\zeta_{(2,1)}(1)) = (1, 0, 1)$, and $y_{(2,1,1)}(\zeta_{(2,1)}(4)) = (4, 2, 2)$. The final tonic chord of this progression has the same values as the opening tonic.

We may wish to translate these $I_{12,7,3}$ elements into equivalent elements of the isomorphic groups, such as frequency ratios (in O_5). Because we are using a unified tonal space that is capable of describing the relationships among tones in many keys, these JI ratios will always be expressed in relation to the identity element C, rather than the tonic pitch. In Chapter 3, we shall use the coordinates of the $V_{2,3,5}$ row vectors to graph chords and progressions on the just-intonation *Tonnetz*. In order to obtain the equivalent $V_{2,3,5}$ row vectors to the $I_{12,7.3}$ elements we derived using ζ_t and y_t, we must apply the function h^{-1} to each of the results that we have just calculated. For example, the $\hat{2}$ in the ii chord, which has $I_{12,7,3}$ value $(4, 2, 1)$, can be translated using the expression $h^{-1}(4, 2, 1) = (-2, 0, 1)$. This, in turn, can be translated into a frequency ratio (relative to C) by applying $v^{-1}(-2, 0, 1) = 2^{-2} \cdot 3^0 \cdot 5^1 = 5/4$. Likewise, the V chord's $\hat{2}$ $(4, 2, 2)$ gives the $V_{2,3,5}$ and N_5 elements $h^{-1}(4, 2, 2) = (-6, 4, 0)$ and $v^{-1}(-6, 4, 0) = 2^{-6} \cdot 3^4 \cdot 5^0 = 81/64$. From these two ratios we can see that, as we noted in Section 2.2, the application of the rules in Table 2.2 cause a theoretical syntonic comma shift between $\hat{2}$ in the ii chord (5/4) and $\hat{2}$ in the V chord ($81/64 = 5/4 \cdot 81/80$). If we had not taken special care to heed Rule 4 in Table 2.2 when performing our calculations, the result would have been an impure theoretical tuning of the supertonic triad. In chromatic progressions, further decisions must be made beyond the simple application of the functions ζ_t and y_t. We shall thus examine one more example of the derivation of scale-based JI from pitch-class integers.

For our second example of the arithmetic calculation of scale-based JI, suppose that a piece in G major contains the abruptly modulatory progression G: I vi E♭: V^7/♭III V$^6_5$ I. To find the scale-based JI values for this progression, our first action is to find the $I_{12,7,3}$ value of the tonic G (pitch class 7). To do this, $t = (0, 0, 0)$ will be used to find the closest G (tonally) to the origin or identity $(0, 0, 0)$. Hence we can write $y_{(0,0,0)}(\zeta_{(0,0)}(7)) = (7, 4, 2)$. To obtain the frequency ratio of this G relative to C, apply the function $v^{-1}(h^{-1}(7, 4, 2)) = 3/2$. Within this key, to obtain the $I_{12,7,3}$ ordered triples of the pitch classes G, (pc 7), B (pc 11), and D (pc 2) of the first chord, we apply $y_{(7,4,2)}(\zeta_{(7,4)}(7)) = (7, 4, 2)$, $y_{(7,4,2)}(\zeta_{(7,4)}(11)) = (11, 6, 3)$, and $y_{(7,4,2)}(\zeta_{(7,4)}(2)) = (2, 1, 1)$. As the submediant triad (pcs 4, 7, and 11) contains $\hat{1}$, we shall calculate it with reference to the same tonic: $y_{(7,4,2)}(\zeta_{(7,4)}(4)) = (4, 2, 1)$, $y_{(7,4,2)}(\zeta_{(7,4)}(7)) = (7, 4, 2)$, and $y_{(7,4,2)}(\zeta_{(7,4)}(11)) = (11, 6, 3)$. As the third chord has dominant function, we shall take the members of this chord from the scale of the key in which it functions. The tonic of the new key is pitch class 6 and, because of Rule 4 in Table 2.2, will take its value from the function $y_{(7,4,2)}(\zeta_{(7,4)}(6)) = (6, 3, 2)$. Hence the $I_{12,7,3}$ values of the members of the third chord (pcs 1, 5, 8, and 11) may be derived using the functions $y_{(6,3,2)}(\zeta_{(6,3)}(1)) = (1, 0, 1)$, $y_{(6,3,2)}(\zeta_{(6,3)}(5)) = (5, 2, 2)$, $y_{(6,3,2)}(\zeta_{(6,3)}(8)) = (8, 4, 3)$, and $y_{(6,3,2)}(\zeta_{(6,3)}(11)) = (11, 6, 3)$. Note the common-tone connection between pc 11 in the second chord and pc 11 in the third chord. As the fourth chord has dominant function in a different key, we shall derive the new scale from the scale most recently used: $y_{(6,3,2)}(\zeta_{(6,3)}(3)) = (3,1,1)$. Note that $\zeta_{(6,3)}(3) = (3,1)$ is D♯, rather than E♭. The members of the fourth chord (pcs 2, 5, 8, and 10) will thus be represented by $y_{(3,1,1)}(\zeta_{(3,1)}(2)) = (2,0,1)$, $y_{(3,1,1)}(\zeta_{(3,1)}(5)) = (5,2,2)$, $y_{(3,1,1)}(\zeta_{(3,1)}(8)) = (8,4,2)$, and $y_{(3,1,1)}(\zeta_{(3,1)}(10)) = (10,5,3)$. In this case, two common tones are held from the previous chord and must thus maintain the same values. The final chord's members (pc 3, 7, and 10) will derive its ordered triples from the same scale as the previous chord: $y_{(3,1,1)}(\zeta_{(3,1)}(3)) = (3,1,1)$, $y_{(3,1,1)}(\zeta_{(3,1)}(7)) = (7,3,2)$, and $y_{(3,1,1)}(\zeta_{(3,1)}(10)) = (10,5,3)$.

Though this mathematical method for generating the coordinates of any pitch or pitch-class in tonal space is necessary for the consistent analysis of extended-tonal music, the pure numerical results are not comprehensible in an immediate and intuitive way. Therefore, in Chapter 3, we shall explore several spatial views of the diatonic/just-intonation system. The products that result from the functions $h^{-1}(y_t(\zeta_t(a)))$ and $y_t(\zeta_t(a))$ will be applied to the just-intonation *Tonnetz* and used in transformational networks. These networks will form the apparatus for displaying analyses in Chapter 5.

CHAPTER 3

SPATIAL GRAPHS AND TRANSFORMATION NETWORKS

3.1 The Just-Intonation *Tonnetz*

The Tuning Lattice

The previous chapter introduced a scale-based method for deciding diatonic spelling and tuning intervals in just intonation. These intervals in JI were represented in pitch-class space as the number of fifths and major thirds that comprise the interval. For example, an ascending minor third or descending major sixth is one fifth "up" from the lower note and one major third "down" from there.[1] This interval can thus be represented as the ordered pair $(1, -1)$.[2] Using this ordered pair, a Cartesian plane, or tuning lattice, can thus be constructed to represent O_5, the set of octave-reduced intervals in 5-limit JI.[3] A minor third away from the origin would thus have x coordinate 1 and y coordinate -1, as shown in Figure 3.1. There is, of course, a lattice point $(1, -1)$ away from any other point on the lattice; thus, the origin here simply represents the first note of the ordered pitch-class interval. If we wish to assign the pitch-class C to the lattice point $(0, 0)$, then the entire lattice can be given note-name labels, as seen in Figure 3.2. Since the lattice in Figure 3.2 extends infinitely in all directions, there is an infinite number of spellings of any of the 12 pitch classes, all separated by $(0, -3) = 128/125$, and an infinite number of instances of each

[1] Since interval directionality breaks down in octave-equivalent pitch-class space, one fifth "up" between pitch classes may also be represented in music as one fourth down.

[2] Recall from Section 2.2 and Section 2.3 that in $V_{2,3,5}$ the first component of the ordered triple represents the number of octaves in the interval; and in $W_{2,3,5}$ this number is dependent upon the second and third components, always reducing the interval to within the span of an octave. The first component can always be generated from the second and third components, and thus can be dropped when it is convenient to do so. To include the first component of the ordered triple on the tuning lattice would require a three-dimensional lattice with x, y, and z axes. This, however, is unnecessary and unwieldy for our purposes. Oettingen (1866) was the first theorist to use this notation for pitch-classes and intervals and apply it to the tuning lattice.

[3] The tuning lattice is familiar to music theorists (who know it as the "*Tonnetz*") from its origins in Euler 1739, 1773, Oettingen 1866, and Riemann 1915, and its recent revival in Hyer 1989, 1995, Mooney 1996, Cohn 1997, 1998a, and the work of many other neo-Riemannian theorists.

spelling separated by $(4, -1) = 81/80$, the syntonic comma. This way of thinking about the *Tonnetz* is demonstrated nicely by the structure of the group $I_{12,7,3}$, where the first component of the ordered triple (containing only the integers mod 12) restricts the number of pitch-classes to 12, but the second and third components (each containing the set of all integers) allow for infinite spellings of these pitch classes and infinite syntonic-comma shifts of each spelling.

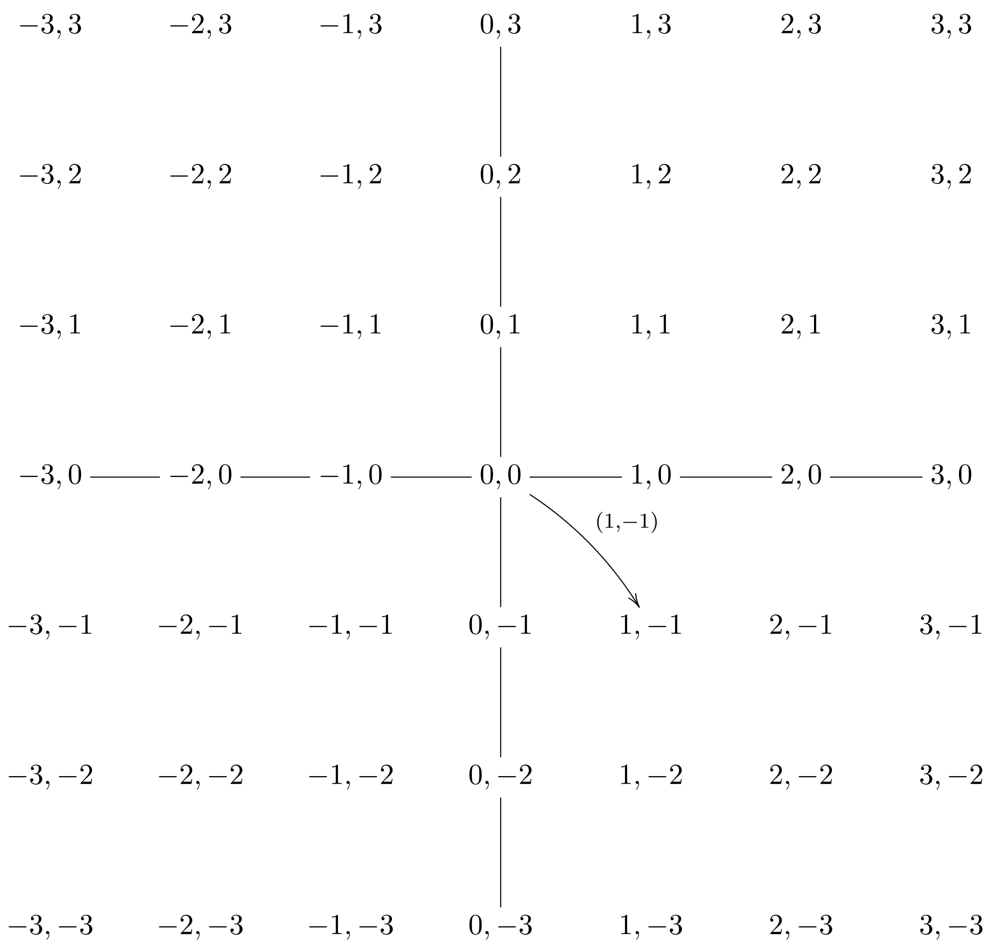


Figure 3.1: Ascending Minor Third/Descending Major Sixth on the Tuning Lattice

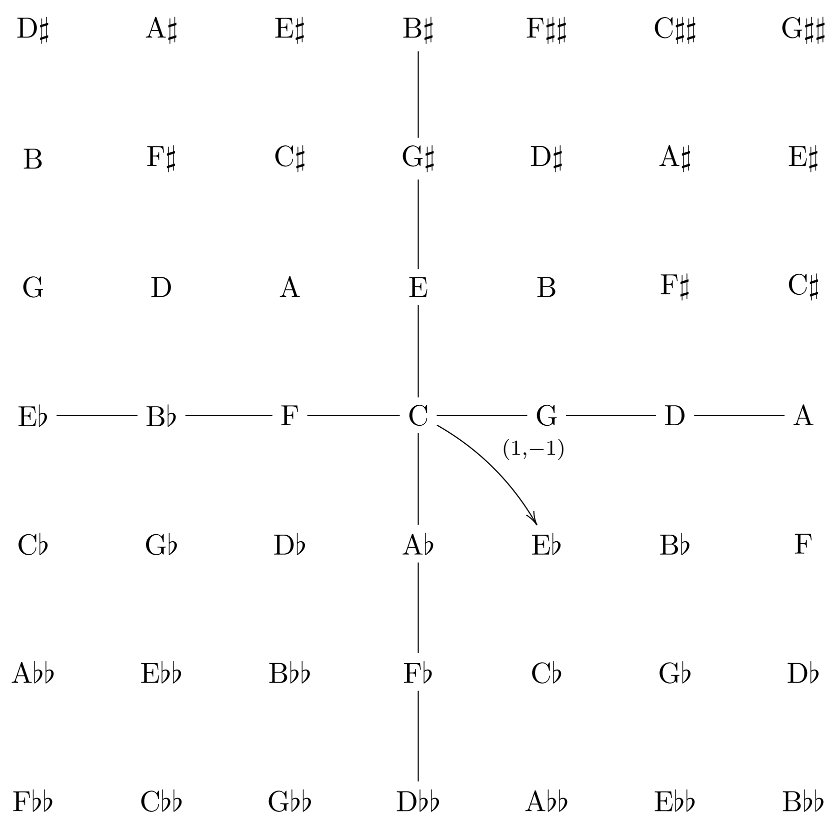


Figure 3.2: Minor Third/Major Sixth on the Tuning Lattice with Letter Names

Temperaments on the Tuning Lattice

In Section 2.2, we examined the use of the matrices

$$H = \begin{bmatrix} 12 & 7 & 3 \\ 19 & 11 & 5 \\ 28 & 16 & 7 \end{bmatrix} \quad \text{and} \quad H^{-1} = \begin{bmatrix} -3 & -1 & 2 \\ 7 & 0 & -3 \\ -4 & 4 & -1 \end{bmatrix}.$$

The columns of H can be thought of as the number of steps that approximate harmonics 2, 3, and 5, in 12-, 7-, and 3-tone equal temperament. The rows of H^{-1} can be thought of as the values in $V_{2,3,5}$ of three intervals.[4] The three intervals given by the rows of H^{-1} are 25/24, 128/125, and 81/80. Three submatrices of H^{-1},

$$\begin{bmatrix} {h^{-1}}_{2,2} & {h^{-1}}_{2,3} \\ {h^{-1}}_{3,2} & {h^{-1}}_{3,3} \end{bmatrix}, \begin{bmatrix} {h^{-1}}_{3,2} & {h^{-1}}_{3,3} \\ {h^{-1}}_{1,2} & {h^{-1}}_{1,3} \end{bmatrix}, \text{and} \begin{bmatrix} {h^{-1}}_{1,2} & {h^{-1}}_{1,3} \\ {h^{-1}}_{2,2} & {h^{-1}}_{2,3} \end{bmatrix},$$

give determinants 12, 7, and 3.[5] Defining these three submatrices may be thought of as assuming octave equivalence (ignoring the first column) and defining the two unison vectors (i.e. the commas that are tempered out) for the (equal) temperament that contains the determinant's number of notes. The unison vectors are the values in $W_{2,3,5}$ of the two rows of the given submatrix. By projecting these two intervals across the tuning lattice, an equal temperament can thus be visually represented as in Figure 3.3. The unison vectors together intersect to form a parallelogram (actually an infinite number of congruent parallelograms) on the tuning lattice, and the area of each section of the lattice defined by the parallelogram is equal to the number of notes in the temperament. Since coordinates on the tuning lattice represent row vectors from $W_{2,3,5}$, Figure 3.3 shows that a 12-tone scale tempers out the two commas $128/125 = (0, -3)$ and $81/80 = (4, -1)$. As we have just noted in Figure 3.2, 128/125 is the difference in 5-limit JI between enharmonically equivalent pitch classes such as C♯ and D♭; and 81/80 (the syntonic comma) is the difference in 5-limit JI between 4 just perfect fifths and a just major third (given by the fifth partial of the harmonic series). The determinant of the 2×2 matrix for each combination of unison vectors given above gives the

[4] Any three distinct intervals would suffice, provided the result is a unimodular matrix. The three ordered triples in H^{-1}, however are particularly useful, as the 12-, 7-, and 3-tone scales generated by the matrix's inverse H are privileged by diatonic scale theorists.

[5] Those who are familiar with matrices and determinants will see that the reason why the determinants of the submatrices coincide with values from H^{-1}'s reciprocal matrix H follows from the definition of a reciprocal matrix (or inversion matrix).

area of the parallelogram and thus the number of tones in the equal temperament that maps (tempers) all tones by those two unison vectors (commas). Any equal temperament can be determined in this way by two commas.[6]

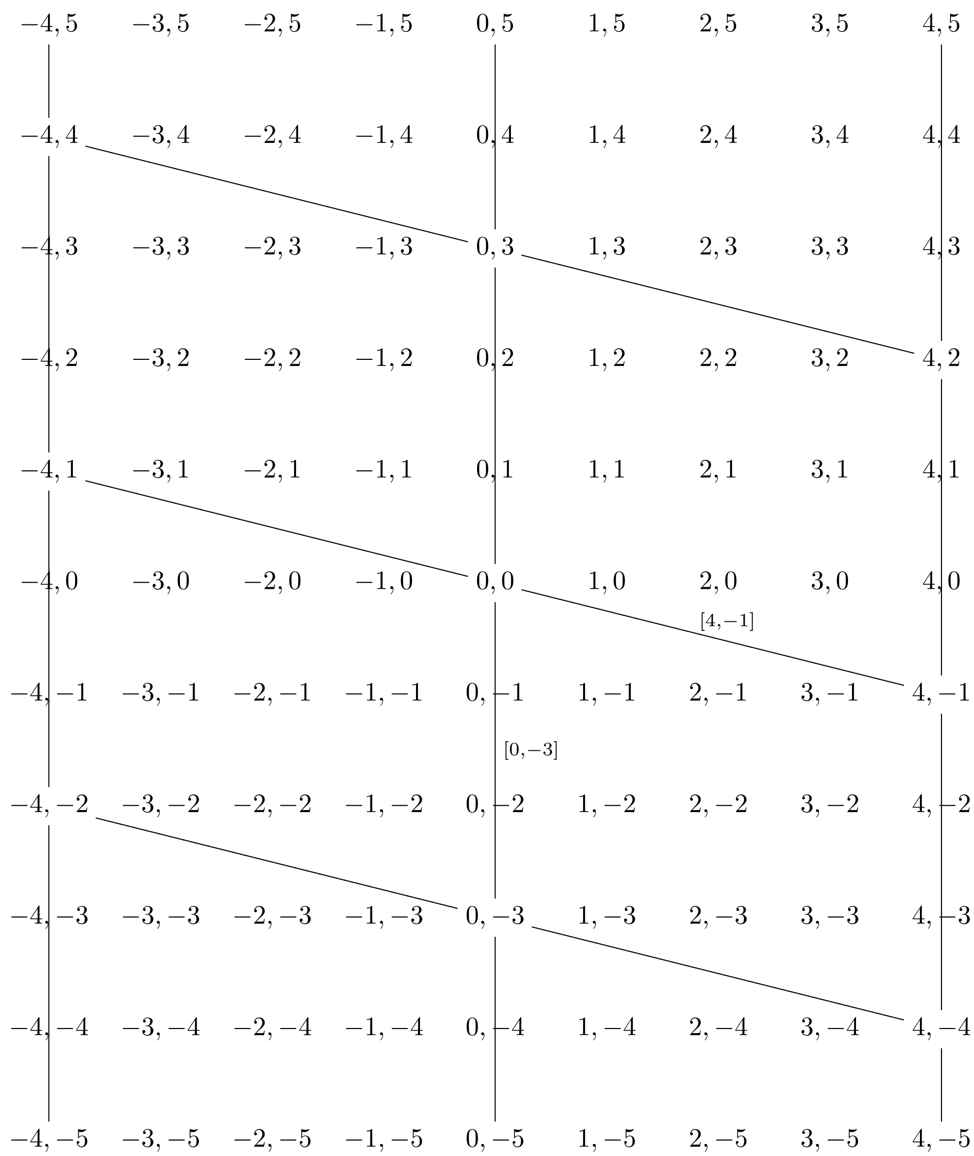


Figure 3.3: 12-Tone Equal Temperament on the Tuning Lattice

[6] Because there are musically nonsensical ways of constructing temperaments using this mechanism, it is important for the theorist to determine *a priori* which commas are theoretically useful in the definition of that n-note scale.

Enharmonic Progressions on the Tuning Lattice

The *Tonnetz* has the ability to illustrate the motion of chord progressions in tonal space. Figure 3.4 shows the shapes that various pitch-class sets will form when each is deployed on the tuning lattice. The chords in this chart are not to be interpreted as bearing any relationship to one another in tonal space. They are given here only so that their shapes will be recognizable when they are seen on a *Tonnetz*. In Figure 3.4, whenever there is more than one distinct tuning of a particular pitch-class set, a Roman numeral analysis of the chord replaces the simple chord-quality label to indicate the chord's function-based tuning. The Roman numeral is also given for shapes where the chord's quality itself (e.g. fully-diminished seventh) implies a distinct tonal function (which may occasionally be left unfulfilled in practice).

Progressions that exhibit "diatonic drift"—that is, they return to an enharmonically respelled chord or key area—will visibly drift when they are mapped on the just-intonation tuning lattice. While many of the possible progressions in chromatic harmony qualify as enharmonic progressions, some composers (e.g. Liszt) favored root motion by intervals that divide the octave evenly, such as major or minor thirds. Figure 3.5 shows the ascending-major-third enharmonic progression from Figure 2.1 on the tuning lattice. Note that the origin $(0, 0)$ represents the pitch class C; the function $y_{(0,0,0)}(\zeta_{0,0}(2)) = (2, 1, 1)$ generates the coordinates $h^{-1}(2, 1, 1) = (2, 0)$ for the initial tonic D; and the $W_{2,3,5}$ values (or Cartesian coordinates) of each chord in the progression are drawn from a different scale (because of non-tertian results in Rule 1 of Table 2.2). While the progression in Figure 3.5 ascends by major thirds, the progression in Figure 3.6 descends by minor thirds. This example also drifts away from the initial chord in tonal space, but the ultimate chord in the progression is a syntonic comma $(4, -4, 1)$ lower than the ultimate chord in Figure 3.5. In terms of diatonic spelling, the chords are the same, but the third terms in their $I_{12,7,3}$ ordered triples are different. Root motion by different intervals that divide the octave evenly thus all end on chords whose roots are different by one or more syntonic commas. Different types of enharmonic progressions therefore wander off in different directions.

As a theoretical entity, this "*Tonnetz* drift" is clearly the result of a dogged adherence to diatonic spelling according to tonal function. Whenever enharmonic progressions appear in tonal music, the perception of the equivalence of the respelled chords or keys implies a conceptual tempered system (viz. a 12-tone temperament). Enharmonic equivalence is thus a prerequisite for the use of this type of progression. Nevertheless, there is a phenomenological confusion inherent in such progressions.[7] While it is easy enough with the progression from

[7]Cohn (1996, 11), when discussing Lewin's (1984) view of enharmonic progressions as "paradoxical and illusory" in Wagner's *Parsifal*, uses the term "vertigo" to describe the effect of such progressions, "which

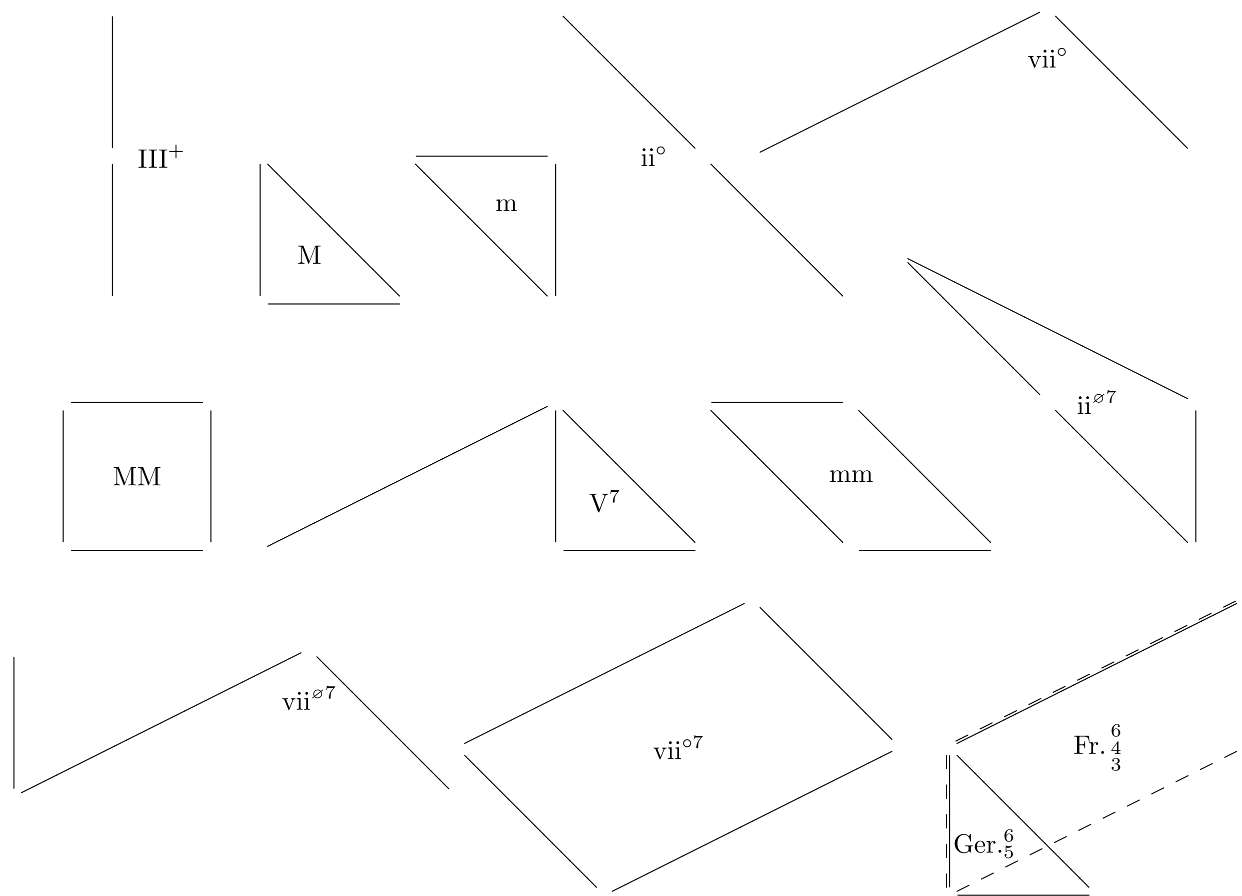


Figure 3.4: Shapes of Various Tonal Chords on the Tuning Lattice

Figure 2.1 to hear the last chord as being the same as the first, it is also easy to imagine how one might lose track of tonic if each of these chords is compositionally expanded into a key area. This perceptual ambiguity between moving away from and returning to tonic allows for intriguing interpretations of this music. The progression through tonal space suggests metaphors of tonal motion as a journey, where diatonic progressions are trips through the well-known paths of the local neighborhood, and where enharmonic progressions are bewildering paths to distant but somehow familiar regions or long journeys that return to a home where something significant has changed. In Section 5.1 the interpretive potential of these metaphors is explored in the analysis of a song by Wolf that features enharmonic progressions.

at once divide their space equally and unequally.... The enharmonic shift cannot be located: it occurs everywhere, and it occurs nowhere."

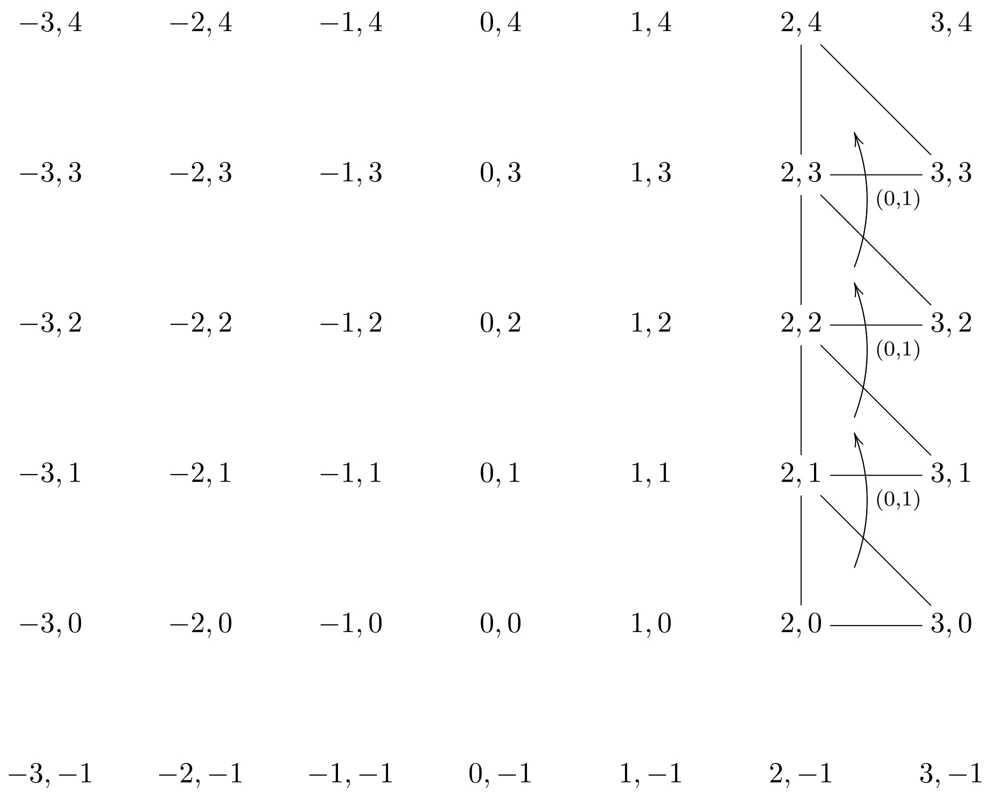


Figure 3.5: The Enharmonic Progression from Figure 2.1 on the Tuning Lattice

3.2 Just-Intonation and Mod-12/7 Transformational Networks

Just-Intonation Networks

Figure 3.7 shows a transformational network in Lewin 1987, 170 (Lewin's Figure 7.9). Lewin uses this graph, which he calls a fundamental-bass network, to relate two passages in Beethoven's Symphony No. 1 in C Major: the openings of the first and third movements. The nodes of the graph are chord roots and transformations are indicated by the frequency ratios associated with the arrows between the nodes. While Lewin is clearly operating in pitch-class space, his transformations do not adhere to the group O_5.[8] We could create a descending octave-reduced 5-limit JI group isomorphic to O_5 (as Lewin has), or we could simply map Lewin's fractions to equivalent ratios in O_5: 4/3, 1, and 5/3. Note that the transformations do not indicate how the quality of the chord that is generated relates to the original chord.

[8] Lewin entirely uses descending intervals between 1/2 and 1 in order to "capture the 'falling' sense of the root progressions".

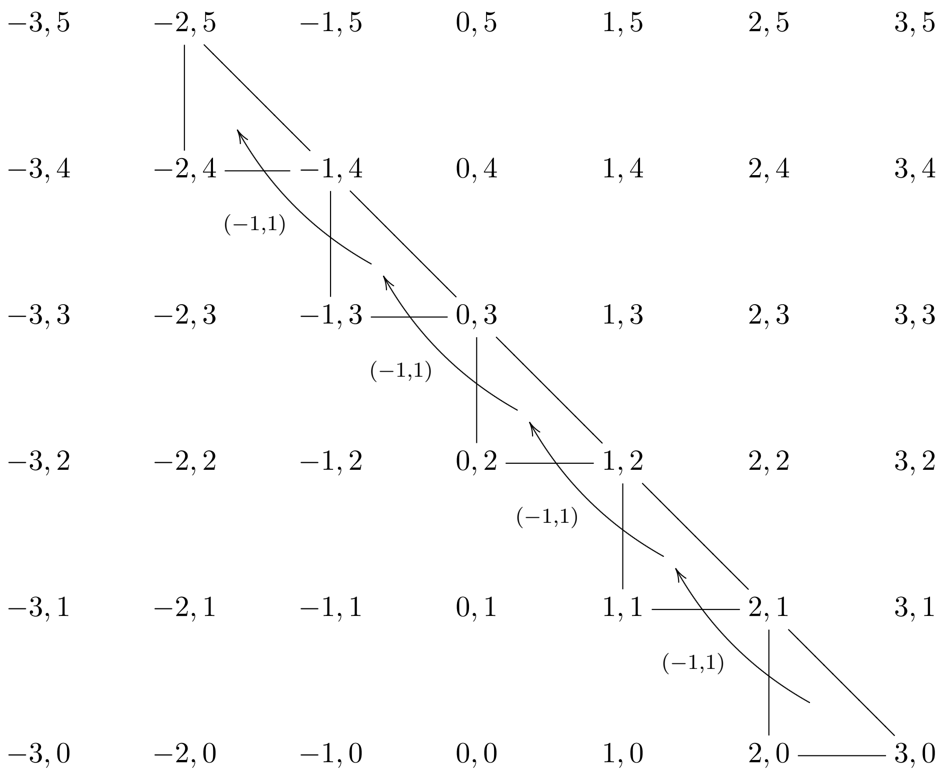


Figure 3.6: An Enharmonic Progression by Minor Thirds on the Tuning Lattice

In other words, the A-minor triad in Figure 3.7 does not result from the transformation 5/6, as this transformation could just as easily generate an A-major triad. This type of network is thus only able to represent root motion, not the content of each chord.[9] This lack of specificity about chord quality is also a feature of the simplest form of mod-12/mod-7 network that we shall explore in this dissertation. For the present analytical purposes, the fundamental bass is sufficient, but features of Hook's unified triadic transformations can be incorporated into the mod-12/mod-7 networks to represent chord quality as well.[10]

The elements of O_5 that indicated intervals of root motion can also be represented as elements from the groups $W_{2,3,5}$ and $I_{12,7,3}$. Figure 3.8 rewrites Lewin's Figure 7.9 using elements from $W_{2,3,5}$ that are structurally equivalent to the O_5 ratios of Lewin's network. While this form of the network is useful for mapping the transformations onto the tuning lattice, this is not the most convenient format for reading the transformations easily.

[9] Lewin later introduces Klang networks to solve this problem, and Hook (2002) builds upon Lewin's Klang networks to create a complete group of unified triadic transformations.

[10] We shall discuss further the sufficiency of the fundamental-bass progression for representing the entire texture of a piece in Section 4.1.

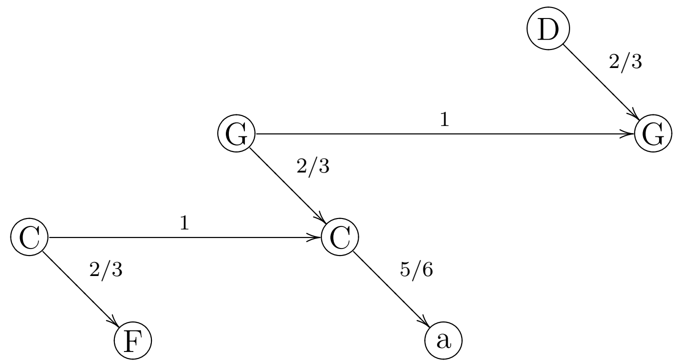


Figure 3.7: A Just-Intonation Transformation Network in Lewin 1987

Figure 3.9 translates the transformations from the previous two networks into elements of $I_{12,7,3}$. In this form of the network, it is easy to find both the generic and specific sizes of the intervals between the roots. The third component of the ordered triple can typically be generated from the first two using the function y, and will thus not always be included in networks of this type. We shall therefore refer to this type of transformational graph as a mod-12/mod-7 network. For the remainder of this chapter we shall explore this type of network further.

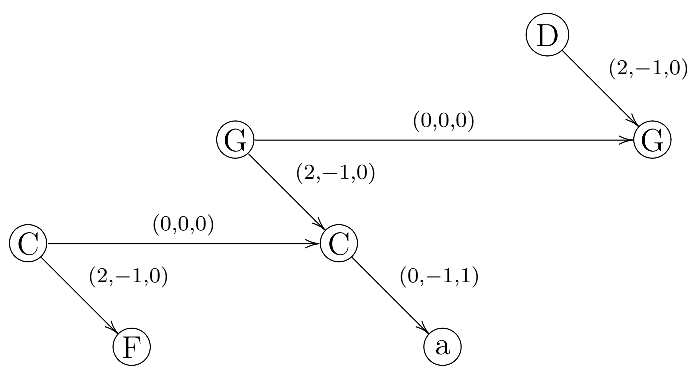


Figure 3.8: Variant of the Just-Intonation Transformation Network in Lewin 1987

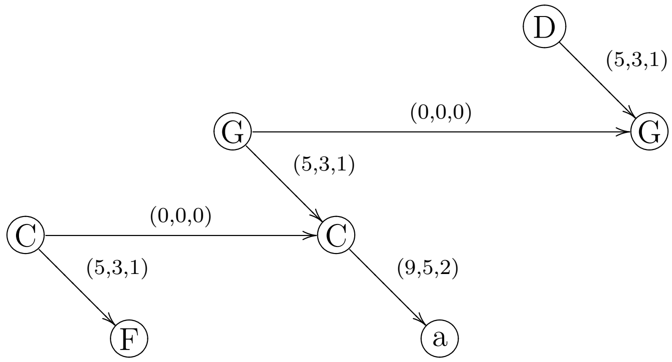


Figure 3.9: Second Variant of the Just-Intonation Transformation Network in Lewin 1987

Mod-12/Mod-7 Networks

As we have just discovered, Lewin's fundamental-bass networks can be relabelled as mod-12/mod-7 networks without losing any substantial information. Just like any transformational network, mod-12/mod-7 networks provide flexible graph structures that can be organized to display certain structural features of the music. For example, the nodes can be organized into harmonic regions so that, when the arrows reflect the temporal order of the music, the graph suggests a journey through these areas. Lewin's fundamental-bass network in Figure 3.7 does this to a certain extent, displaying dominant-tonic relationships as diagonals in one direction, and displaying the tonic-dominant relationships of each successive two-chord progression as diagonals in the opposite direction. In the next section we shall explore further possibilities with regard to organizing graphs to reflect tonal hierarchies.

Mod-12/mod-7 networks can also serve to highlight some of the peculiarities of JI-based diatonic spelling, such as diatonic drift. As an example, Figure 3.11 graphs the enharmonic progression from Figure 2.1 and Figure 3.5. For the most part, the arrows indicate the temporal order of the chords in the progression. The arrow from C♯♯ back to D, however, indicates a retrospective comparison between the newly achieved chord and the original tonic triad. If the music were to continue on from here, the network would also have a second arrow from the C♯♯ to whatever chord root follows it in the score. If a piece of music were to drift further diatonically, the chord roots would become increasingly burdened by n-tuple sharps or flats and increasingly unwieldy to interpret. We shall thus henceforth use the convention that chord roots will always be spelled as they appear in the score. Figure 3.2 rewrites Figure 3.11 using this convention to simplify the spelling only in the network's nodes. While this procedure simplifies root note names such as C♯♯, it creates a discrepancy between the spelling of the nodes and the spellings implied by the transformations. As enharmonic respellings are always expressed by either $(0, 6)$ or $(0, 1)$, the transformations accompanying the arrows will clearly show enharmonic progressions without the need to respell the roots' note names as well. In other words, only the transformations accompanying the network's arrows are to be used for interpreting the diatonic spelling or JI tuning of the chords in the piece. To accomplish this, we shall create a function α to generate the ordered-pair value of the first node. Let a be an integer representing the letter name of the pitch class, where C = 0, D = 1, ..., B = 6. Further, let b be an integer value corresponding to the pitch class's chromatic alteration, where $\ldots, \flat\flat = -2, \flat = -1, \natural = 0, \sharp = 1, \sharp\sharp = 2, \ldots$. Given a and b, we can define the function $\alpha(a, b) = ((((((2 \cdot a) \bmod 7) \cdot 7) \bmod 12) + b) \bmod 12, a)$. The correctly spelled ordered pair of any other node can be generated by taking the sum (mod-12 and mod-7) of this first node's ordered pair and the ordered pairs of all intervening transformational arrows along a single path. In order to translate that node's ordered pair back into its correctly spelled note name, apply the inverse function $\alpha'(c, d) = (d, ((((7 \cdot c) \bmod 12) - (((2 \cdot d + 1) \bmod 7) - 1) \cdot 7 + 5) \bmod 12) - 5)$. While these functions clarify and formalize the interpretation of mod-7 transformations, they can be bypassed entirely once familiarity with the mod-12/mod-7 notation for diatonic spelling is achieved.

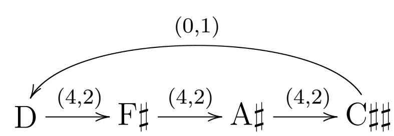


Figure 3.10: Mod-12/Mod-7 Network Showing Progression in Figure 2.1

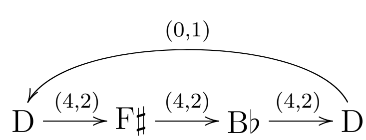


Figure 3.11: Mod-12/Mod-7 Network Simplifying Root Spelling in Figure 3.11

3.3 Prolongational Transformational Networks

Networks for Schenkerian Analyses

While mod-12/mod-7 networks are useful for obtaining a clear perspective of enharmonic progressions, they may also aid the visualizion of tonal hierarchies. Figure 3.12 reproduces a transformation network in Lewin 1987, 216 (Lewin's figure 9.16) that is designed to encapsulate many of the basic features of an outer-voice Schenkerian sketch. The nodes of Lewin's graph contain a chord symbol, the soprano scale degree supported by that harmony, and the hierarchical level of the chord in the analysis. The suggestive names of Lewin's transformations are almost descriptive enough to completely decipher how they act on the chords. The transformation PROJ moves a chord into the next higher or lower analytical level on the graph without changing anything but the third component of the node's ordered triple. SUST and N+ operate on the second component of the ordered triple, indicating the sustaining of the *Urlinie* scale degree or an upper neighbor.[11] DOM transposes a triad up by 5 semitones (or down by 7 semitones), and likewise SUBD transposes a triad down by 5 semitones (e.g. DOM maps F major to B♭ major, and SUBD reverses the mapping). REL and PAR are neo-Riemannian operations that change the mode of a triad by means of a contextual inversion. REL produces a triad representing the relative major or minor key (e.g. REL maps G major to E minor and vice versa), and PAR produces a triad representing the parallel major or minor mode (e.g. PAR maps E minor to E major and vice versa). Based on Lewin's model, it would be possible create similar networks describing the Schenkerian composing-out of entire pieces. One could thus translate back and forth between a graph of this type and rudimentary Schenkerian notation on a staff. Figure 3.13 performs this translation of Lewin's network in Figure 3.12 into a Schenkerian sketch. While the notation in Figure 3.12 accurately represents Lewin's network, using open noteheads and stems to distinguish between analytical levels, a Schenkerian sketch that uses idiomatic notation might include extra features such as an eighth-note flag on the bass note E♭.[12]

Although mod-12/mod-7 transformations typically are not full-fledged klang networks as is Lewin's graph, fundamental-bass progressions alone can be representative of harmonic prolongation. Figure 3.14 replicates Lewin's graph as a mod-12/mod-7 prolongation network. Each node of the network is represented as an ordered pair where the components are

[11] One could thus imagine other possible operations such as DESC, CS, N-, P+, and P-, for *Urlinie* descents to the next scale degree, consonant skips, lower neighbors, and *Züge* (passing/linear motion).

[12] This brings up the question of whether these networks are in any way a useful replacement for Schenkerian notation. I believe that Schenkerian notation offers the possibility of more sophistication, subtlety, and complexity than a prolongational network. The networks can, however, serve a pedagogical purpose or offer a tool for clarifying one's thoughts about what pitches exist on what levels of the piece's hierarchical analysis.

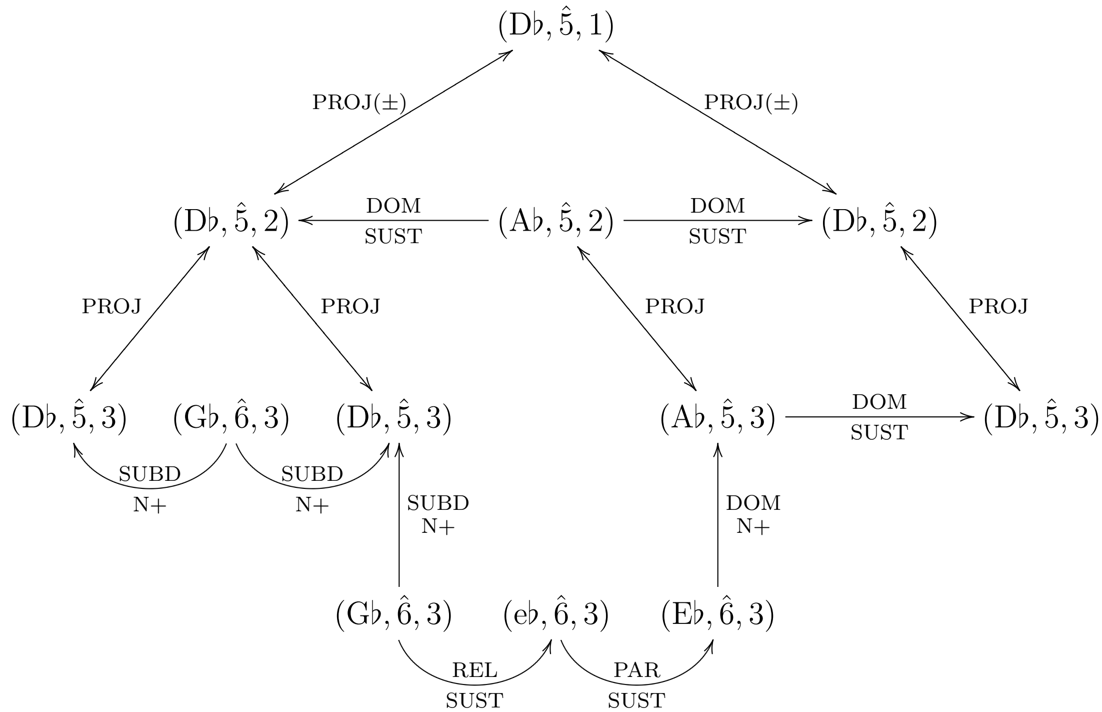


Figure 3.12: A Prolongational Transformation Network in Lewin 1987

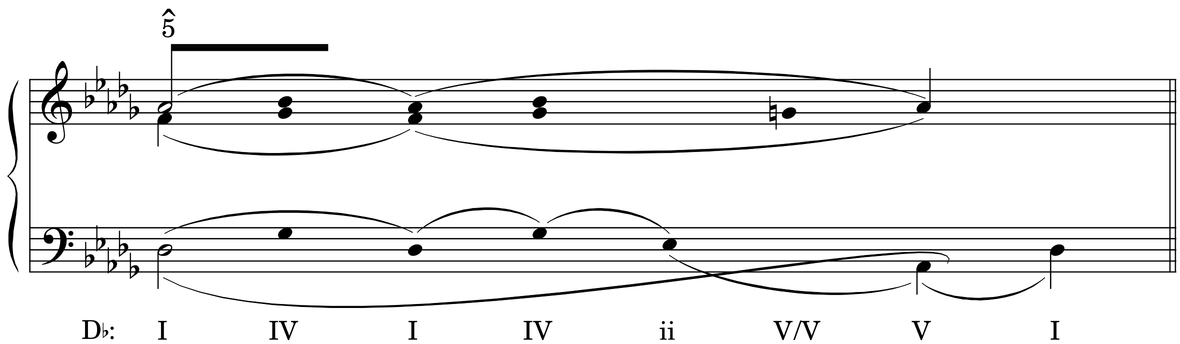


Figure 3.13: Lewin's Prolongational Transformation Network as a Schenkerian Sketch

fundamental bass (and quality), prolongational level. The transformations are no longer represented as ordered pairs in this network, but instead as ordered triples. The third component of the ordered triple is not the 3-tone component of the $I_{12,7,3}$ element, but rather an operation on the level component of the nodes. All transformations that move to members of the same level have 0 as their third component, while transformations that project a chord into a shallower hierarchical level or refer back to a deeper level have +1 or −1 as their third components, respectively. For purposes of clarity, in future graphs, this third component of the transformation may be left out in networks where the third component's value is always obvious from context.

Certainly Lewin's network has more musical specificity and intuitive appeal than the mod-7 network in Figure 3.14. In the present case, the only real advantage of the mod-12/mod-7 transformations is that they encapsulate information about a work's tonal scale-degree functions that is not found in the mod-12 perspective embodied by Lewin's transformations.[13] In general, mod-12/mod-7 networks make use of a larger group of transformations than Lewin's networks and thus may address more complicated music. For example, to represent a simple chord progression such as major V to minor i a Lewinian network would have to combine two operations, PAR(DOM), while the mod-7 transformation is simply $(5, 3)$. A different example, where V moves to iv, would require three of Lewin's operations, DOM(PAR(DOM)), while the mod-12/mod-7 operation is merely $(10, 6)$.[14] While transformation networks are often trivial for simple chord progressions, and while Lewin's networks are preferable for the prolongational analysis of traditional tonal music, mod-12/mod-7 networks may be more useful for representing enharmonic progressions or prolongational hierarchies of non-traditional progressions with precision. We shall therefore explore both of these applications. First, the next section will demonstrate the use of prolongational networks to refine the transformational graphing of enharmonic progressions. Then, in Chapter 5 we shall use mod-12/mod-7 prolongational networks to explore both the analysis of music that features enharmonic progressions, and the analysis of works that resist a strict Schenkerian analysis.

Enharmonic Progressions in Mod-12/Mod-7 Prolongational Networks

Figure 3.15 represents Figure 2.1 using a mod-12/7 prolongational network. Recall that we have already used Figure 3.5 and Figure 3.11 to elucidate the enharmonic progression in Figure 2.1. Figure 3.11 uses a transformational arrow from the final chord in the progression back to the first chord to show the retrospective connection between the two diatonically different chords. While the perceptual metaphor implied by this transformational network indeed captures Lewin's sense of the paradoxical quality of this progression (inducing Cohn's

[13]Q.v. Cohn's (1998a) quote in Section 1.2 on page 4. This certainly is no reason to abandon Lewin's graph in favor of the mod-12/mod-7 graph.

[14]In this case, DOM(PAR(DOM)) captures an aural sense of the progression rather poorly, especially if the progression simply resolves the dominant deceptively.

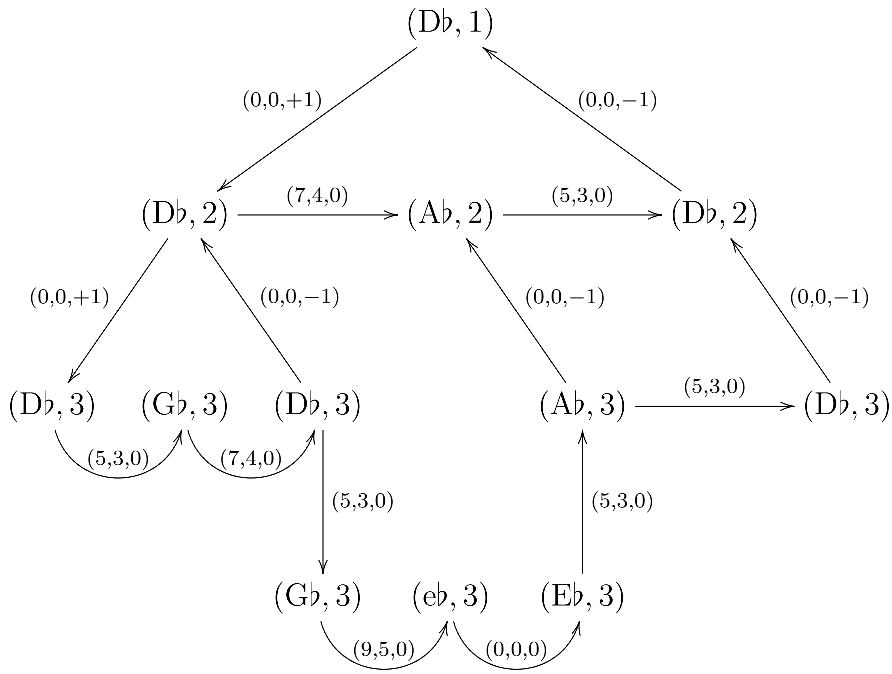


Figure 3.14: Lewin's Prolongational Transformation Network as a Mod-12/Mod-7 Graph

"vertigo"),[15] a prolongational view, on the other hand, compares the final chord to a prolonged chord on a more background level, thus providing a more accurate theoretical assessment of the reason for the confusion of diatonic degrees. Specifically, if one is to hear the progression as prolonging the first D major chord, there must be a mental diatonic correction of the final chord, which is shown in the arrow pointing back to the first level.[16]

Enharmonic progressions are not the only oddities of chromatic harmony that are illuminated by a transformational perspective. In Section 5.2, we shall examine the graph structure that results from the use of mod-12/7 networks on a work that ends in a different key from its opening. Other examples of unconventional tonal usage may also result in unusual network structures as well. It is important to note, however, that these transformational networks could possibly create formally correct but musically meaningless structures. To create a prolongational graph, one must simply be able to determine the roots of the chords in the music. While chords that resist traditional root-finding methods are an impediment to using these networks, they are also an impediment to the analysis of music as a prolongational

[15]Q.v. footnote 7.

[16]The reader is reminded of our convention to use convenience spellings in the nodes, since the transformations define the theoretical spellings.

structure. The next chapter seeks first to establish a single method for determining chord roots, and then, more importantly, to define the terms within which prolongation may be posited.

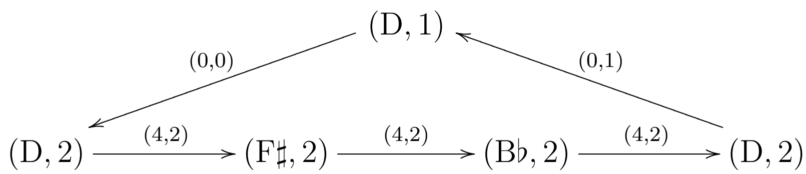


Figure 3.15: Mod-12/Mod-7 Prolongational Network Showing Progression in Figure 2.1

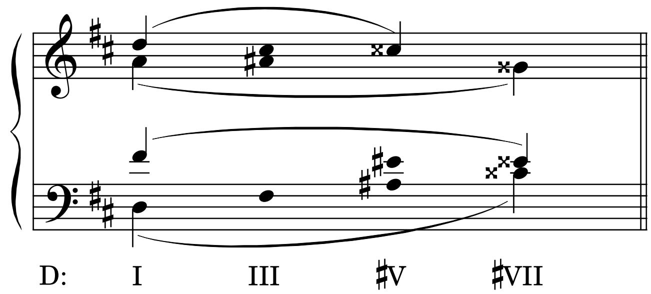


Figure 3.16: Prolongational Network in Figure 3.15 as a Schenkerian Sketch

CHAPTER 4

DETERMINING CHORDAL SALIENCE IN POST-FUNCTIONAL MUSIC

In Chapter 2, we built a foundation for the diatonic interpretation of tonal music. While the diatonic spelling procedures given in Table 2.2 return consistent results provided an accurate key analysis of the work (including all tonicizations and modulations), post-functional music may thwart key analysis through ambiguity of chord function or of diatonic scale. The first and most important impediment to prolongational analysis of post-tonal literature is thus ambiguity of key center. Indeed, diatonic spelling based on a tonal center is an important element in the hierarchization of the music's pitch-class content, since the diatonic scale itself represents a hierarchy around a tonal center. Diatonic scale members tend to be structural, and chromatic notes tend to be transient. Furthermore, diatonic spelling allows for the discovery of passing and neighboring motions in diatonic pitch-class space.[1] Pervasively fluent passages are helpful in determining how one might go about hearing a passage as prolonging one harmony, but pervasive fluency does not determine what chords are structural or how one may find prolongations. This chapter addresses how one might decide what is structural when the music defies certain norms of tonal harmony. In order to display the resulting passage as a mod-12/mod-7 prolongational network, one must reduce the chord progression to a fundamental-bass pattern. It will thus be helpful to have a consistent way of representing the root of a chord.

4.1 Finding Chord Roots

Hindemith (1942) provides a comprehensive method for determining the root of any chord based on his dogmatic view that there is an acoustical necessity for all chords to have a single root, that the idea of chord inversion is somewhat misguided, and that determining

[1] In Section 1.3 we examined Jones's (2002) theory of prolongation in chromatic harmony through the analysis of pervasive fluency. Pervasively fluent passages (PFPs) contain passing and neighboring connections among all chord members of both the initiating and terminating chords.

chord roots by stacking a chord in thirds should be replaced by a more comprehensive intervallic analysis. For our present purposes, we need not make such definitive statements regarding the structure of a chord or the relative prominence of its members.[2] When creating a mod-7 transformational network, the analyst must select the part of the chord that is to participate in a fundamental-bass progression. For this purpose consistency of result is more useful than "correctness" of root. In other words, while it would be ideal for our analytical system to allow the freedom to choose a system of contrapuntal representation that suits one's own theoretical bias,[3] arguably the most accessible transformational graphs are those that display chord symbols in a widely accepted and highly legible format. Because popular chord symbols, along with our system of diatonic spelling, have the potential for representing the same contrapuntal information as a thoroughbass reduction or Jones's diatonic lattice, I have made an arbitrary choice to use that system of chord analysis. Fundamental bass analysis does not emphasize the contrapuntal structure of the progressions it represents—rather, it emphasizes the vertical structures in the music. Because harmonic analysis, though downplaying horizontal concerns, still retains contrapuntal information, the analyst is still obligated to translate the chord notation into a format where the voice leading concerns that substantiate prolongation can be addressed. For this reason, a sketch using traditional notation for linear analysis appears along with every prolongational network in this dissertation.

There is nevertheless a theoretical preference for the fundamental bass note at the heart of the theory constructed in Chapter 2. While I would certainly hesitate to reiterate Hindemith's assertion that every chord needs to have a single definitive root,[4] in Chapter 2 it was important for the purpose of determining harmonic function and diatonic spelling to rely upon the chord tone that deviates the least from a key's center in tonal space. The apparent root bias of the theory itself therefore also should not be understood to stand in the way of the contrapuntal view that should underpin our system of prolongational analysis. In fact, the hierarchization of scale degree functions that was introduced in Chapter 2 and that forms the basis of our method of root analysis also may contribute to one's interpretation of the contrapuntal structure. Indeed, this perspective that tonal function is both a harmonic and a contrapuntal concern is supported by Harrison 1994 and Agmon 1996b.

[2]While Hindemith's writing is perhaps too prescriptive, and his method fails to account for diatonic means of defining the function of the scale degrees within a chord, his perspective on the matter is still worthwhile. Indeed, the ability of a chord to retain its root when the bass note changes does tend to break down beyond seventh chords. For further arguments for Hindemith's chord-group system and root analysis procedures, see Harrison 2004.

[3]For example, my personal bias would lead me to represent a contrapuntal texture with figured bass rather than fundamental bass.

[4]Kaminsky (2004) discusses Hindemith's assertion and the possibility of more than one perceptible root in polychords. In this section we shall also encounter some chords with ambiguous roots.

While there should certainly be some interaction between the harmonic and contrapuntal aspects of prolongation, the determination of a root will not play a role in deciding whether a passage can support prolongation. We shall discover, however, that those structures for which the root-finding mechanism falters might also be those that lead away from clear tonal function.[5] Unless some specific feature of the music disambiguates the function of such chords, no chord with ambiguous root should be placed in a "structural" position on a prolongational network.[6] This single constraint is the extent to which root finding can determine chordal salience.[7] With these less ambitious aims for a theory of chord roots in mind, we shall now examine some simple schemes that provide a fundamental bass note that may be used on a transformational network.

Tertian, Quartal, and Overtone Chords

The simplest method of root finding is to rearrange the pitch classes in a chord so that they proceed by thirds (or by fifth if a note is missing from the interval projection). The root is always the bottom note in the stack of ascending thirds. While this succeeds in giving consistent results, as Hindemith points out, it may not always reflect the functional root of a chord. Harrison (2004) gives the excerpt seen in Figure 4.1 as an example of this contradiction and its possible derivation from musical gestures such as the ending of Figure 4.2. Stacking the "add 6" chord from Figure 4.1 in thirds gives the "added" note as the root.[8] Chords such as $\mathrm{C}^{\mathrm{add}\ 6}$ may therefore appear in mod-12/mod-7 networks in cases where there is reasonable support for such an analysis. Another impediment to this method of root determination is the ambiguity of the root of certain tertian chords such as 3–12 (048) and 4–28 (0369) where the diatonic spelling of the chord alone determines its root. Because, as we have seen in Chapter 2, diatonic spelling is predicated upon key analysis, a key analysis is therefore necessary before one can determine the root of such chords.

[5] Harrison (2004) agrees in this regard, contending that hearing a piece as tonal, regardless of the acoustical complexity of the tonic (and other structural chords), is dependent upon the analyst's perception of a "rooted tonic".

[6] We shall see some typical examples of chord ambiguity and possible contexts for their tonal clarification at the end of this section.

[7] As we shall discover in Section 4.2, certain features of a chord that help in determining its root can play a role in support of one's case for the structural importance of the chord. These are but a few of the many factors, however, that should play a role in the decision of chordal salience.

[8] Santa (1999, 44–45) also presents an example of this phenomenon, showing a major seventh chord functioning as a minor "add 6" chord. This raises the question of under what circumstances the "root" of set class 4–26 (0358), 4–20 (0158), or 4–27a (0258) is in fact the first note of the ascending tertian ordering. A root-finding system that gives Harrison's (1994) "bases" as roots may perhaps satisfy this ambiguity between tertian root-finding and tonal function. Fortunately, this debate does not require resolution here, as a fundamental-bass progression that follows the tertian root gives consistent results that allow for arguably the most legible transformation graphs.

Figure 4.1: Weill, “Die Moritat von Mackie Messer”, *Die Dreigroschenoper* (1928), mm. 1–16

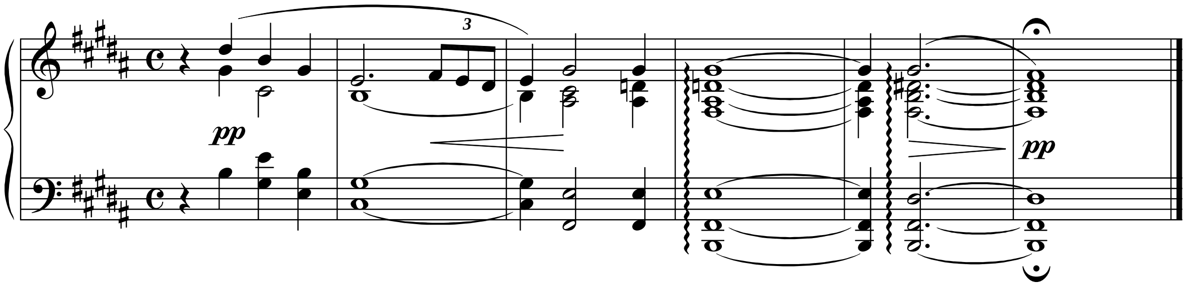


Figure 4.2: Liszt, Ballade No. 2 in B Minor (1853), Ending

Of course not all chords are organized in thirds, and sonorities that are generated from other interval projections may even appear within the context of a tonal piece. For the sake of consistency, we shall use a similar root determination procedure for these chords, if they are clearly organized as interval projections in pitch space. Hindemith’s root-finding method prefers fifths over fourths, and his method will give different results than this perspective, which favors the bottom note of interval projections of any sort. In other words, as long as the chord is constructed using a single interval, the most consistent fundamental bass note is the lowest note in the sequence. This applies to quartal and quintal harmonies, semitonal clusters, and whole-tone scale segments.[9] Fifths seem to be the upper limit of the interval projection root-finding model, as sixth projections can be expressed as tertian harmonies. Further, if the chord projects only a generic interval, such as a diatonic pattern of major and minor seconds or thirds, key analysis should be used to decide the spelling of the chord. Because tonal considerations may also often play a role in deciding the root of a chord, a systematic procedure for deriving chord members' tonal functions may be devised. Using just intonation as a reference for tonal function, we can determine a "root representative" for any tonally unambiguous chord.[10]

Root Representatives

Because we made use of the harmonic series in defining the tonal space outlined in Chapter 2, we can also use it as a resource for root-finding.[11] Table 4.1 outlines the procedure for determining a chord's root representative. Because the method given in Table 2.2 sometimes gives rather complex ratios, the first step of the process allows us to replace some of these more complex 3-limit and 5-limit intervals with simpler ratios that have higher prime limits. All of the ratios in scale-based JI where $r > 32$ that are not listed in Table 4.1 are chromatic intervals, for which we shall assign simpler ratios after examining some examples below. Each of these replacement intervals has an acoustical basis, but they are not practical JI tunings for common-practice tonal music.[12] Steps 1–3 of Table 4.1 help us to determine a ratio representation of the chord in lowest terms, and in the fourth step we then perform an octave reduction to determine the lowest partial of the overtone series from which each pitch class can be said to arise. In the list given in the fifth step, the intentional placement of the number five before three corrects for the overtone series's bias toward major thirds over minor thirds.[13] This root-finding algorithm returns the same results as Hindemith's in

[9]The use of whole-tone scale segments within the context of functional harmony may in fact clarify a functional root. We shall discuss the function and diatonic spelling of the whole-tone scale further when we examine Table 4.1.

[10]Hindemith (1942, 101) first used the term "root representative" for the note that may be used as the root in diatonically ambiguous chords such as augmented triads and diminished seventh chords. I have appropriated the term in a more general sense here, using it to refer to Hindemith's procedure of deriving a fundamental tone from a conglomeration of notes that imply a single harmonic series. The method defined in the next section (in Table 4.1) is reminiscent of the acoustical basis of Hindemith's procedure, but far less comprehensive. Harrison (2004) also invokes Hindemith's "root representatives", in support of the conclusion that pitch-space interval projections take their bottom note as a root.

[11]Väisälä (1999 and 2002) supports his own prolongational view of post-tonal music by examining how chords relate to the harmonic series.

[12]For more on the use of 7-limit consonances to replace 5-limit tunings of seventh chords, see Regener 1975 and Monzo. Further, footnote 20 from Section 1.3 provides some guidance for the introduction of 7-limit consonances into the JI system outlined in Chapter 2.

[13]When we ignore this alteration, the results of this root-finding method seem to favor a theoretical view that is different from harmonic dualism (which is also rooted in JI theory) and more aligned with the view

most cases, but it is not as comprehensive.[14] While it provides definitive results for all tonal harmonies (including extended tertian chords and diatonic scale segments), the results for post-tonal sonorities are far more ambiguous. In this section, we shall engage in the useful (if somewhat tedious) exercise of examining several instances of tertian, non-tertian, and post-tonal chords and using the method in Table 4.1 to designate a root representative for each.

Table 4.1: Procedure for Finding a Chord's "Root Representative"

1. Using the method outlined in Chapter 2, determine the N_5 ratios (5-limit JI tunings) of all members of the chord. If possible, rewrite the chord as a multiple ratio (r_0:r_1:r_2: ... :r_n), where $r < 32$. The ratio 32:27 may be replaced with 7:6, 64:45 may be replaced with 7:5, 16:9 may be replaced with 7:4, and 225:128 may be replaced with 7:4. Extended tertian chords may use even higher prime numbers.
2. If the chord still cannot be represented as a multiple ratio (e.g. 4:5:6:7) or its use is forbidden by restrictions in the previous step, use Figure 4.3 to find the multiple-ratio representation of the chord with the lowest prime limit. The chord may not be respelled to fit a particular set of partials that are spelled differently in the figure.
3. Reduce the ratio so that there is no common factor among all overtones. For example, 12:15:18 has 3 as a common factor, and thus reduces to 4:5:6.
4. Divide each of the integers that represent chord members by 2 until none of the integers is divisible by 2. For example, the overtone chord 10:12:15:18 (representing a minor seventh chord) reduces to 5:3:15:9
5. To decide the root, choose the part of the reduced chord that appears earliest in the following ordered list: 1, 5, 3, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31.

One common tonal sonority for which a root representative needs to be determined is the German augmented-sixth chord (or the Italian sixth, which is a subset of the German sixth chord).[15] According to the first step in Table 4.1, the method in Table 2.2 is to be used

of minor and diminished triads as elided 5–6 motions from the more prominent diatonic triads. Jones (2004) uses a different method for correcting the harmonic series's bias against minor thirds. When using the area that a chord covers on the *Tonnetz* to measure its instability, Jones alters the angle between the axes on the *Tonnetz* to 60° so that the distance between notes a major third apart and a perfect fifth appart is the same as the distance between notes a minor third apart.

[14] The method described here also resonates with Harrison's (2004) idea of "root amplification".

[15] The fact that German sixths sound like root-position dominant sevenths yet appear to be in first inversion creates considerable ambiguity in the identity of the chord's root. The root-finding method in Table 4.1 is not intended to resolve this ambiguity, but rather to give a consistent result for the purposes of creating a

Figure 4.3: Diatonic Spelling of the Harmonic Series

to determine the tuning of the chord.[16] Suppose that the augmented-sixth chord appears in the key of C major. Given the tonic pitch class of C major, $t = (0, 0, 0)$, the function $v^{-1}(h^{-1}(y_t(z_t(8))))$ returns the ratio 8/5, $v^{-1}(h^{-1}(y_t(z_t(0))))$ gives 1/1, $v^{-1}(h^{-1}(y_t(z_t(3))))$ = 6/5, and $v^{-1}(h^{-1}(y_t(z_t(6))))$ is 45/32. The ratios between adjacent chord members are thus 5/4, 6/5, and 75/64. There is no multiple ratio (r_0:r_1:r_2: … :r_n) where $r < 32$ that represents this chord. However, the ratio between pc 8 and pc 6 is 225:128, which may be replaced with 7:4. Therefore the multiple-ratio representation of this chord is 4:5:6:7. The fourth step in the procedure reduces the ratio to 1:5:3:7 by dividing out all multiples of two, and the final step gives pc 8 as the root representative. The procedure also returns the multiple ratio 4:5:6:7 for dominant-seventh chords, since they contain the interval 32:27.

While the Italian sixth chord will simply be 4:5:7 and have the same root representative as the German sixth, we should investigate the French sixth to determine which chord tone will serve as its representative on mod-7 networks. Like the other two augmented-sixth chords, the French sixth contains pitch-class ratios (relative to 1/1 as tonic) 8/5, 1/1, and 45/32. The French sixth contains $\hat{2}$ as well, which according to the function $v^{-1}(h^{-1}(y_{(0,0,0)}(z_{(0,0)}(2))))$ is 9/8 of tonic. This chord also contains the ratio 225:128, which can be replaced by 7:4. This gives the multiple ratio 32:40:45:56, which does not satisfy the condition $r < 32$. We thus cannot skip step 2 in Table 4.1 in this case. Hence we must find the lowest prime-limit multiple-ratio representation of the chord. In Figure 4.3 we can find a French sixth one octave above the 4:5:6:7 German sixth, at 8:10:11:14 (32:40:45:56 becomes 32:40:44:56 with common factor 4). In the final two steps of the process, this ratio reduces to 1:5:11:7, and (in

fundamental-bass progression. Indeed, it is my contention that many chords in tonal harmony do not have a single definitive root, even though we shall be assigning a root to represent these chords in mod-7 networks.

[16]Much of the tuning procedure in Table 2.2 was encapsulated into the mathematical functions that were introduced in Section 2.4. We may thus use these functions to generate the tuning of each pitch class in the chord.

C major) pc 8 thus represents the fundamental bass of this chord as well. What, then, of the conception of the French sixth as a sort of V^4_3 with a lowered fifth? Certainly this method denies the tertian root of this chord its status, even in cases where the chord is used in root position as a dominant. Nevertheless, consistent results are returned for all augmented-sixth chords; and if it seems contrary to a reasonable analysis of the music to call the altered note the root of an inflected dominant seventh, the label $D^7_{\flat 5}$ is not forbidden in mod-7 networks.

As we saw in Chapter 2 there are two distinct tunings of the half-diminished seventh chord in scale-based JI. Neither tuning (25:30:36:45 or 45:54:64:80), however, satisfies the conditions of step 1 in Table 4.1. Because the $\text{vii}^{\varnothing 7}$ tuning of this chord contains the interval 32:27 (64:54 reduced to lowest terms), we can replace it with 7:6 and use the outside interval from the $\text{ii}^{\varnothing 7}$ tuning (9:5) to arrive at the multiple ratio 5:6:7:9 (45:54:64:80 becomes 45:54:63:81 with common factor 9). The representative note of this chord (once it is reduced to 5:3:7:9) is thus its traditional tertian root. In step 5 of Table 4.1, what looks like a sequence of odd numbers has 5 and 3 reversed. This change of ordering results in returning the traditional root of any tertian chord that begins with a 6:5 minor third.

The fully-diminished seventh chord, of course, will take its representative from its tertian root, once its diatonic spelling has been established. It will nevertheless be worthwhile to determine if the procedure in Table 4.1 returns the tertian root of this chord as its representative. The tuning of the members of the $\text{vii}^{\circ 7}$ chord as is returned by the tuning method in Table 2.2 is (relative to 1:1 as tonic) 15:8, 9:8, 4:3, and 8:5. While this tuning of the chord also does not form a multiple ratio with $r < 32$, we can, as we did with the half-diminished seventh chord, substitute 7:6 for the 32:27 interval in the chord. This gives 5:6:7 for the first three members of the chord. Although the fourth member of the half-diminished seventh chord is 9:5 from the first note, the fourth member of the fully-diminished seventh chord is 128:75 from the first note, and thus must also be replaced. In Figure 4.3 one octave above 5:6:7, we can find a correctly spelled replacement for the final pitch class in the chord. The chord will thus be represented by 10:12:14:17, and can be reduced to 5:3:7:17, with its first note as its root representative.

Now that we have used our JI-based root-finding scheme to define the root representatives of most typical tonal chords, we can now turn to the examination of some non-tertian and diatonically ambiguous chords that may appear in extended-tonal music. First, let us examine some non-tertian chords, beginning with a three-note quartal chord. The set 3–9 (027) can be represented by the multiple ratio 9:12:16, or, when stacked in fifths, 4:6:9. Because these reduce to 9:3:1 and 1:3:9 respectively, our root-finding procedure always returns the same root for this set class, preferring stacked fifths over stacked fourths, in contradiction to our convention of using the bass note of any interval projection. This fact supports my claim that the procedure in Table 4.1 is similar to Hindemith's root-finding method. The same situation obtains for larger quartal/quintal chords (e.g. 4–23 (0257) has multiple ratios 27:36:48:64 and 8:12:18:27). Chords may also be built by stacking up seconds. The diatonic set 7–35 (013568A) as a multiple ratio tuning is 24:27:30:32:36:40:45. Because $r > 32$ here, we must increase the prime limit to find its representative. Without the $r < 32$ requirement, the root representative of the set would indicate that the Lydian mode is the primary organization ("root position") of the sonority. The only diatonic sets in Figure 4.3 are 16:18:20:21:24:27:30 and 16:18:19:21:24:26:29. The first of the two options has the lower prime limit, which implies Ionian as the set's "root position".

While the diatonic collection and chords built with stacked fourths and fifths are not diatonically ambiguous, other interval projections are, including augmented triads, diminished seventh chords, and clusters of semitones and whole tones. Further, many chords that are not interval projections are spelled differently based on the scale degree on which they are built.[17] In all of these cases, the diatonic spelling of the collection must be taken from the musical context (key analysis) before it can be located on the overtone series. For example, a three-note semitone cluster could be an expression of $\flat\hat{7}$ $\natural\hat{7}$ $\hat{1}$, spelled as an augmented unison and a minor second, or $\hat{7}$ $\hat{1}$ $\flat\hat{2}$, spelled as two minor seconds. In both cases, since $r > 32$, we must find the lowest prime-limit instance of the set with the correct diatonic spelling in the overtone series. With a prime limit of 11 in both cases, these two multiple ratios are 21:22:24 and 11:12:13, respectively, giving $\hat{1}$ as the representative in each case. With whole-tone clusters, care must be taken to replicate the spelling of scale degrees in Table 2.1, including correct placement of the diminished third. For example, the melodic minor scale has five out of six notes of the whole-tone scale between $\flat\hat{3}$ and $\hat{7}$. Adding $\flat\hat{2}$ from the Phrygian mode gives a complete whole tone scale, with a diminished third between $\hat{7}$ and $\flat\hat{2}$. The multiple ratio 13:14:16:18:20:22 has two potential diminished thirds (augmented sixths) in it: 16:14 and 13:11 (26:22).[18] This ambiguity and the exaggerated wideness of the diminished third 26:22 suggests that using the whole-tone scale in the next octave of the overtone series might be more successful. With the multiple ratio 17:19:21:24:27:30, none of the major seconds exceeds 9:8 and the single diminished third 17:15 is fairly close to the 256:225 5-limit diminished third in Table 2.1. The root representative of this selection of overtones is $\hat{5}$ of the "melodic Phrygian" scale, implying a potential dominant function

[17] Recall that Table 2.2 included a proviso in Rule 1 that tertian chords may override any non-tertian spellings in Table 2.1. In fact, any diatonically unambiguous chord may also override any conflicting spellings in Table 2.1, while diatonically ambiguous chords may not. I enumerate the diatonically unambiguous chords near the end of this section.

[18] Recall the use of 8:10:11:14 for the French sixth chord, implying a diminished third between 14 and 16.

for the sonority.[19] However, this also implies that music that does not define the key center diatonically, but rather restricts itself to the whole-tone collection, using emphasisis to define centricity cannot be analyzed using this method.

Further, there are pitch-class sets for which diatonic spelling is not sufficient for finding a root representative. One example of this is 3–3 (014). There are three possible spellings for this set, each involving a chromatic interval, all of which have a potential tonal use. Expressed based on one possible tonal use of the spelling (using scale degrees), the three spellings are $\flat\hat{3}$ $\natural\hat{3}$ $\hat{5}$ (containing an augmented unison), $\hat{5}$ $\flat\hat{6}$ $\natural\hat{7}$ (containing an augmented second), and $\natural\hat{7}$ $\hat{1}$ $\flat\hat{3}$ (containing a diminished fourth). In each case there are multiple locations on the overtone series where the chord can be found as spelled, often giving different answers about the set's representative note. Because the overtone series is useful only for providing definitive tunings for diatonic intervals, our method of root-finding has difficulty with chords that contain chromatic intervals with little inherent diatonic context. Restricting the tuning of chromatic intervals to particular overtone combinations allows for a definitive decision among these conflicting solutions. Augmented unisons should always be 20:19, augmented seconds are 15:13 or 20:17, depending on the other notes in the chord, diminished thirds are 8:7 or 17:15, diminished fourths are 13:10, and augmented fourths are 7:5. The (014) chord and all others with uncontextualized chromatic intervals will nevertheless remain in a class of "tonally ambiguous" chords that we shall bar from analytical placement in a structural position on a prolongational network unless the musical context defines the use of that chord. An example of such a context-defined chord in this category is the diminished-seventh chord. When used in tonal contexts, the diatonic spelling of the chord defines its root and the location and tuning of the chromatic interval in it (A2). Thus, provided a diatonic framework for its use, a diminished seventh chord may be prolonged. Likewise, the set 3–3 (014), once contextually given its tonal function, may also be prolonged.

While the division between diatonically ambiguous and tonally ambiguous chords is hazy, the set of diatonically unambiguous chords is well defined. The only diatonically unambiguous set classes (including their inversions) are given in Table 4.2.[20] All other chords have more than one (theoretically) potential function in diatonic space (using the scale degrees found in Table 2.1) when spelled in different ways. For example, the major-minor seventh chord 4–27b (0368) may be spelled either $\hat{5}$, $\hat{7}$, $\hat{2}$, $\hat{4}$ or $\flat\hat{6}$, $\hat{1}$, $\flat\hat{3}$, $\sharp\hat{4}$. Further,

[19] To enhance this inherent dominant function, this collection could be organized in thirds to form $V^{\flat 13}_{\substack{9\\ \flat 5}}$.

[20] Santa (1999) and Jones (2002) both also categorize the set classes according to their ability to support alternative spellings. My list differs somewhat from theirs because my purpose is to suggest the circumstances under which we can interpret a chord as a diatonic-scale-based harmonic entity regardless of its participation in linear chromaticism. In this way, we may even extrapolate diatonic scales from chromatic occurrences of diatonically unambiguous chords. Rule 4 from Table 2.2 accomplishes this task.

the minor-major seventh chord 4–19a (0148) may be spelled either $\hat{1}$, $\flat\hat{3}$, $\hat{5}$, $\hat{7}$ or $\flat\hat{6}$, $\hat{7}$, $\flat\hat{3}$, $\hat{5}$.[21] The only diatonically unambiguous chords are thus those that can be represented within the uninflected diatonic collection and that contain one or more perfect fifths and no tritones. Unambiguous chords have a single root representative, regardless of bass note or spacing. Interval projections that are not diatonically ambiguous are represented by the bottom note when they are arranged using the generating interval. Diatonically ambiguous chords require key analysis before selection of the root representative is possible. Though this distinction is useful for root finding, it does not necessarily distinguish chords' function or salience in a musical texture. Even diatonically unambiguous chords can be used in unusual ways as transient chords. For example, 3–7 (025), while normally an incomplete minor-seventh chord with pitch class 2 as its root representative, can appear as a triad with pitch class 0 as its root and two non-chord tones that will resolve to pitch classes 4 (or 3) and 7. It is therefore important for the root-finding method presented here to be reserved for sonorities whose tonal function is not clarified by the musical context. Note, however, that unusual uses of these diatonically unambiguous chords do not negate their ability to contribute to a sense of key center.

Table 4.3 codifies the procedure we have been building in this chapter for performing analysis using mod-7 networks. The analytical apparatus rests on key analysis, and the ordering of the first three steps is designed to ensure a consistent reading of tonal centers for all tonal and extended-tonal literature. The next section will elaborate upon the last step in the analysis procedure, and provide a set of criteria from which one may make decisions about possible prolongations.

4.2 Finding Structural Chords

Perhaps the most important part of prolongational analysis is the determination of structural and transient verticalities. Even in tonal music, it can be difficult to render decisions about what is structural and what is not. For example, Wagner (1995, 166–168) cites a disagreement between Forte and Gilbert (1982, 115) and Laskowski (1984, 116) about what chords are structural in the opening bars of the second movement Mozart's Piano

[21] At least two chords listed in Table 4.2 do in fact have the potential for multiple interpretation. Specifically, 3–7 (025) and 4–26 (0358) may be respelled as non-traditional augmented-sixth chords. (See Harrison 1995 for further treatment of the more extravagant augmented-sixth chords of chromatic harmony.) In the rare and easily identifiable cases where music uses these chords in this way, the rules in Table 2.2 allow the analyst to retain the non-tertian spelling. The typical examples of multiple interpretation using augmented sixths, however, also hinge upon the reinterpretation of a tritone. For this reason, we shall continue to consider members of interval class 2 as stable diatonic entities.

Table 4.2: Diatonically Unambiguous Sets

2–5	(05)	(perfect fourth/fifth)
3–4	(015)	(incomplete major-seventh chord)
3–7	(025)	(incomplete minor-seventh chord)
3–9	(027)	(two stacked fifths)
3–11	(037)	(minor/major triad)
4–10	(0235)	(minor tetrachord)
4–11	(0135)	(major/Phrygian tetrachord)
4–14	(0237)	(incomplete minor ninth chord)
4–20	(0158)	(major seventh chord)
4–22	(0247)	(incomplete major ninth chord)
4–23	(0257)	(three stacked fifths)
4–26	(0358)	(minor seventh chord)
5–23	(02357)	(minor/major pentachord)
5–27	(01358)	(major/minor ninth chord)
5–35	(02479)	(pentatonic scale)
6–32	(024579)	(Guidonian hexachord)

Sonata in D Major, K. 311, shown in Figure 4.4.[22] Forte's and Gilbert's reading appears in Figure 4.4(b), while Laskowski's analysis is shown in Figure 4.4(c). While Wagner presents a third, more nuanced reading of the passage, the decision of what is structural in this and many other excerpts is not straightforward. While tonal music offers a well-defined harmonic syntax to guide the analyst, in music where this syntax breaks down such decisions become far more difficult and we must rely on subjectivities to a greater extent. Keeping this in mind, I shall provide guidelines to help determine what is and is not structural. These considerations are not intended to replace an analyst's intuition and musical experience, but rather they are designed to help guide one's attention to certain features of the music. Among the considerations for asserting chordal salience that we shall discuss are functional connections, motivic connections, and acoustical arguments for chordal salience.

Acoustical Stability

In the previous section we explored an acoustical basis for chord organization with the intent of finding a systematic way of determining fundamental bass notes. We can also use the relationships of harmonic structures with these systems of acoustical organization as

[22]Wagner also presents Salzer's (1952, Vol. 2, p. 50, Ex. 183) more ambiguous reading of the passage along with these two clearly contradictory analyses.

Table 4.3: Analysis Procedure for Creating Mod-7 Networks

1. Root analysis: Defer determination of the roots of diatonically ambiguous chords until after step 2.

2. Key analysis: Use chord roots and qualities and fundamental bass patterns along with one's knowledge of harmonic practice to determine the key for every timepoint in the piece.

   (a) All diminished-seventh chords and major-minor seventh chords imply tonicizations of some key (perhaps simply the prevailing key).

   (b) Secondary dominants and diminished-seventh chords require a change of key for as long as the chord is sounding.

   (c) Secondary progressions and modulations require a change of key through the last dominant in the secondary key or the last chord before the pivot chord or direct modulation.

   (d) Some pieces may begin and end in different keys, even if the analyst will eventually want to read the background of the piece as an auxiliary cadence. (For more on the analysis of directional tonality, see Section 5.2.)

3. Diatonic spelling: Spell every pitch in the piece based on the procedure given in Table 2.2. Use the key-based diatonic spelling to make final decisions on ambiguous-root chords.

4. Pervasive fluency: Make decisions about possible prolongations and use pervasive fluency to see how the prolongational passages may be heard as transient.

one of the considerations in chords' relative salience. For example, holding all other factors constant, a chord that is organized as an interval projection in pitch space affords more weight from its acoustical stability than a chord that is only an interval projection by virtue of its constituent pitch classes. Consider, for instance, the spacing of the fifth projection in Figure 4.5(a) as opposed to a more muddled spacing of the same five pitch classes in Figure 4.5(b). Because many other factors play into chordal salience, however, we cannot assert that acoustically more stable chords will necessarily play a more structural role in a prolongational analysis than less stable chords. I can certainly imagine instances in tonal music where the most convincing analysis shows the exact opposite situation: for example, a tonic chord in first inversion appearing as a passing chord between a ii^6_5 chord and a $\mathrm{vii}^{\circ 6}_{5}$ chord, or the consonant triads that result as passing chords in an omnibus progression. Regardless, let us now examine a method for measuring the relative acoustical stability of

Figure 4.4: Two Conflicting Readings of Mozart, Sonata in D Major, K. 311, II, mm. 1–4

various chords, keeping in mind that this is only one of several considerations that may support a prolongational reading.

While we shall consider interval projections in pitch space to be acoustically stable, we can determine relative degrees of stability using the same harmonic model that we used for choosing root representatives.[23] Given a harmonic-series representation of a chord as derived from the method in Table 4.1, the prime limit of its multiple ratio provides a measure of

[23]Recall that, for the purpose of finding a root representative, Table 4.1 and Figure 4.3 may be used to conceptualize chords as overtones above a single fundamental.

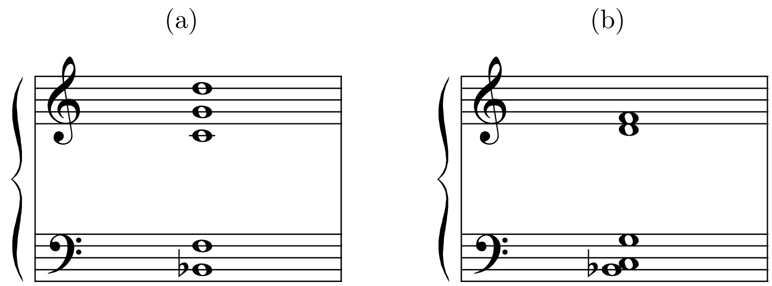


Figure 4.5: Two Voicings of an Interval Projection with Different Stability

the chord's relative degree of acoustical stability. The lower the prime limit, the more stable the chord (regardless of cardinality). For example, holding all other factors constant, this perspective deems a major triad with prime limit 5 as more stable than a dominant-seventh chord with prime limit 7. Likewise, when used as chords, the diatonic collection (16:18:20:21:24:27:30) with prime limit 7 is more acoustically stable than the whole-tone collection (17:19:21:24:27:30) with prime limit 19. Interestingly, both the diminished-seventh chord and the augmented triad are more unstable than the entire diatonic collection according to this method of measuring stability. The overtone method also supports the notion that projections of fourths and fifths are indeed highly acoustically stable, as these chords all have prime limit 3 (or 5 or 7 when they are tuned from the JI diatonic scale rather than from pure fifths).

We can therefore use a chord's prime limit as an integer value defining the instability of any chord. Holding other factors constant, the higher the prime limit is, the more acoustically unstable the chord is. Thus, major and minor triads have an instability value of 5, along with major and minor seventh chords, while dominant and half-diminished seventh chords have instability value 7, and fully diminished seventh chords (along with dominant flat-ninth chords) have instability value 19. Unless a chord is diatonically unambiguous,[24] one must determine the key and spell the chord according to Table 2.2 before applying the overtone method. In the sense that the tonal context (the key) determines the spelling and function of these diatonically ambiguous chords, the tonal context also defines their relative stability.[25] The most likely candidates for prolongation are the diatonically unambiguous

[24] In the previous section we defined the diatonically unambiguous sets as any set that can be represented within the uninflected diatonic collection containing one or more perfect fifths and no tritones.

[25] A chord's stability, then, is judged based on its internal relationships in the tonal space defined in Chapters 2 and 3. Jones (2004) provides another useful chordal-stability model measured using tonal space.

chords. Regardless of their prime limit, these should therefore be considered to have the lowest possible instability value (i.e. 1). This is the only exception to taking the prime limit as instability measure.[26]

One final acoustically defined measure of a chord's stability is the scale degree upon which it is built. In tonal music, triads built on $\hat{1}$ and $\hat{5}$ typically take on structural roles and rarely prolong triads built on other scale degrees. In extended tonality, a composer could establish an alternative hierarchy of roots while maintaining the same principles. A simple but decidedly radical example of this is Lendvai's (1971) "axis system" for the analysis of Bartók's music, where a prolongational analysis may favor tertian chords built on $\hat{1}$ and $\sharp\hat{4}/\flat\hat{5}$, followed by $\hat{5}$ and $\sharp\hat{1}/\flat\hat{2}$, etc. While "axis tonality" provides a useful analogy to elucidate the idea of alternative scale-degree hierarchies, a more straightforward reorganization of scale-degree functions will perhaps carry more weight toward the assertion of prolongation. Since, this idea of the relative stability of root scale degrees is at least as much a functional measure of chordal salience as it is an acoustical measure of stability, we shall now explore tonal function as a basis for asserting prolongation.

Functional Equivalence

In tonal analysis, the connection between two structural chords can indicate either prolongation (meaning the chords share the same function) or tonal motion (meaning the chords are connected by arpeggiation or passing motion). When tonal function is not clear, however, these types of prolongations are much more difficult to assert. It may be wise, if also restrictive, to look with skepticism at prolongation connecting chords of two different types in post-common-practice tonality. Therefore, the analyst must address the issue of whether and how harmonic function is established in the music.

Since the question of how the music creates a context for the perception of chord function is perhaps the most important the analyst must address, this will form the first of the list of criteria for evaluating prolongation that appears at the end of this section. We shall address the absence of harmonic function as an impediment to prolongation further once we have considered the interaction of motive and tonal hierarchy.

[26] This may actually be an unnecessary provision, as all diatonically unambiguous sets already have low prime limits.

Motivic Significance

Perhaps the most controversial aspect of any argument for prolongation is the role that motives play in the assertion of structural tones or chords and prolongation.[27] Santa (1999) advocates the use of motives to help define structural points in his associational model of levels for post-tonal diatonic music. As Santa points out, however, this is an entirely different kind of hierarchy than a system of prolongational levels. Since our prolongational model for extended tonality requires a basis in traditional prolongational theory, our model must accept a more nuanced view of the interaction of motive and prolongation. There are two basic roles a motive or theme can play in the perception of prolongation.

First, the return of a motive can distract from a sense of prolongation. This view can support the idea of the recapitulation in a binary form as an interruption rather than a simple progression from dominant back to tonic. The idea of motivic return as interruption also encourages the analyst to support claims for prolongation with tonal and harmonic structures rather than motivic returns. For this reason in the prolongational model outlined here, motivic significance, like acoustical stability, must form only part of any claim for prolongation.

Second, the return of a motive in special cases may form part of an argument for the termination of a prolongational span. In the absence of the return of a functionally or acoustically stable chord, the return of motivically significant material at the termination of a contrapuntally fluent passage may signal the end of a prolongational span. Once again, such an assertion of prolongation would be aided by the presence of other contributing factors.

Because the factors that contribute to prolongation may interact in different and unique ways, the analyst must weigh the contributions of each factor and make an informed subjective decision. Table 4.4 offers a rubric for evaluating a potential prolongation. In association with Table 4.3 one can use Table 4.4 to create a mod-7 prolongational network of a tonal or neo-tonal work. In the next section, we shall examine some features of a piece that may hamper prolongational analysis.

[27]For background on this issue see Burkhart (1978). My view of the use of motives within prolongational analysis agrees with Cohn's (1992) reevaluation of the issue.

Table 4.4: Considerations for Judging a Single Prolongational Span

1. Does the passage begin and end on the same harmony? If not, what allows the beginning and terminating chords to be heard as having the same tonal function? Provided a context for hearing function, some fluent progressions may connect the end of one prolongational span to the beginning of another span with a different harmonic function.

2. Are the beginning and ending chords at least as acoustically stable as the intervening chords? If not, how are they distinguished from the prolonging chords as being structural?

3. Does the passage exhibit contrapuntal fluency? Do passing and neighboring motions connect the prolonged chords, at least in the pitch-class counterpoint? If not, is there another means of connecting the prolonged chords, e.g. through chord patterning (ABCBA, ABCDABCDA, etc.)?

4. Account for all "chromatic notes" that come from outside the main collection of the passage or work. They must be non-chord tones or part of a modulation or tonicization.

5. How do motives and themes contribute to or detract from the prolongations?

4.3 Impediments to Prolongation

The previous section outlined a largely intuitive procedure for finding prolongational passages, culminating in the rubric in Table 4.4 that allows the analyst to evaluate the previously outlined criteria for prolongation as part of a nuanced view of their interaction. Because of the subjective nature of the endeavor of evaluating prolongation, there are few caveats in the list of criteria for asserting prolongation. There are nevertheless factors which may detract from a prolongational view of music on the fringes of tonal practice. An investigation of these factors will help us to know early on in the analytical process whether a tonal analysis would be fruitful.

Chord Variety

Traditional tonal music uses a relatively small number of set classes (around ten, depending on the style of music).[28] There is often a wider variety of set-class types present in atonal music, making it difficult for the listener to determine any hierarchy among the harmonies. Because tonal music has evolved over a considerable amount of time, and is a culturally learned style, it is easy for experienced listeners to make sense of the complex relationships of the ten tonal sonorities. Whereas much of the extended-tonal repertoire uses a larger number of different chord types, in many cases these are built upon the traditional tonal sonorities in the form of extended-tertian chords, polychords, or color chords (triads with added timbral notes).[29] While it is still possible to find prolongational structures in music with an entirely new harmonic vocabulary, a wide variety of new chords with no clear hierarchy among them may form an impediment to finding prolongation. One tool an analyst may thus utilize in order to contribute to the evaluation of a prolongational analysis of extended-tonal or post-tonal music is a list of distinct set classes in the work and a list of distinct set classes found in structural positions of the work.

Absence of Harmonic Function

We have examined how prolongation may be posited without traditional harmonic function, but the complete absence of a centric pitch class and some rudimentary harmonic

[28]The ten primary chords of common-practice tonal music are 3–10 (036) (the diminished triad), 3–11a (037) (the minor triad), 3–11b (047) (the major triad), 3–12 (048) (the augmented triad), 4–20 (0158) (the major seventh chord), 4–25 (0268) (the French augmented-sixth chord), 4–26 (0358) (the minor seventh chord), 4–27a (0258) (the half-diminished seventh chord), 4–27b (0368) (the dominant seventh chord or German augmented-sixth chord), and 4–28 (0369) (the fully diminished seventh chord). Others can usually be accounted for as arising from leaving out pitch classes from one of the ten sets, using non-chord tones, or extending the chord upward by thirds.

[29]Harrison (2004) provides a useful summary of these extensions of triadic harmony.

syntax can succeed in destroying any sense of prolongation. Let us now enumerate some basic kinds of tonal functions that contribute to a sense of tonal hierarchy. First, scale degree functions, as defined by combining the tonal space posited in Chapters 2 and 3 with traditional fundamental-bass harmonic theory, are essential to establishing harmonic function. A second type of tonal function is a chord's degree of acoustical stability, or, in more traditional terms, a chord's relative consonance or dissonance. Further, a note's function can be distinguished as being diatonic or chromatic, and a passage can take on a structural or transient role based on the presence of diatonic or chromatic harmonies (among other factors). Other rudimentary harmonic functions we have discussed include the prominence of two or three different fundamental-bass notes serving as "poles", such as Bartók's "axis harmony". One last type of harmonic function that follows from the idea of a "tonal polarity" is harmonic dualism. Such a set of harmonic functions could be fulfilled by a synthetic system of upward and downward interval projections, such as quartal harmonies, or by replacing the normal tonic/phonic inversional pair with some other inversionally asymmetrical set class. Not all of these primary tonal functions need to be present for prolongational analysis, but the less one can find discernable harmonic function, the less likely one will be able to find prolongation.

Absence of Reference Collection

Most of the music addressed by our theory of prolongation in extended-tonality has a diatonic scale at its core, no matter how chromatic the musical texture is. Without some "diatonic" reference, the distinctions that define many of the rudimentary tonal functions we have just enumerated disappear. Scales other than the diatonic major/minor system that we have been studying may also be able to create quasi-tonal prolongational structures. Or they may exist within a largely tonal piece as "digression" prolongations of the sections that return to a more stable tonal language. Without a scale on which the music is based, the possibility of finding prolongation quickly diminishes. It may thus be wise to take a conservative stance regarding the reference collections that we shall accept as admitting of prolongation. Straus (1987) was skeptical about prolongation in non-tertian harmony as well. The clouding of the distinction between linear and harmonic elements (non-tertian harmony) is certainly a possible impediment to prolongation, but a much more devastating blow to the possibility of a hierarchical structure is non-diatonic music. Without a reference collection, there is no chance of creating harmonic/melodic distinctions to satisfy any of Straus's criteria for prolongation.

In the next chapter we shall use the conditions for prolongation in Table 4.4 to aid in the analysis of two extended-tonal works by Ravel. First, we shall examine works where the prolongation itself is not problematic in order to show how transformational analysis of enharmonic progressions and directional tonality can support the interpretation of dramatic subtext in two Wolf songs.[30] Then, the procedure for diatonic interpretation in Chapter 2 will serve to create a mod-7 network of the famous chromatic opening to Wagner's *Tristan und Isolde*. Finally, our procedure for finding prolongation will contribute to an analysis of two Ravel piano works, one displaying non-tertian functional harmony, and the other displaying non-tertian harmony lacking traditional tonal function.

[30]We shall also explore how both enharmonic progressions and directional tonality are nevertheless excursions from common tonal practice.

CHAPTER 5

ANALYTICAL EXAMPLES

In the previous four chapters, we have examined how diatonic scale theory, mod-12 and mod-7 group theory, transformational networks, and basic paradigms of prolongational hearing may work together to form a tonal view of the problematic passages of late nineteenth-century and early twentieth-century tonal music. In this chapter, we shall apply these theoretical approaches in various combinations to musical examples that feature extended tonal techniques. First, we shall explore how the diatonic spelling of tonal music according to scale-degree function breaks down in music that involves enharmonic progressions. This confusion among normally autonomous scale degrees will offer a chance to find dramatic subtext in a Hugo Wolf *lied*.[1] Then, we shall discuss how a tonal perspective may be applied in various ways to a second Wolf song that begins and ends in different keys. In such cases, the prolongational and transformational approaches both offer different and useful tools to help the analyst to interpret the music. A third application of the theories examined in this dissertation for the analysis of nineteenth-century chromatic music is the elucidation of highly chromatic passages where the harmonies transcend traditional tonal function. Using the spelling guidelines in Table 2.2, we shall find two viable diatonic interpretations of the famous opening of Wagner's *Tristan und Isolde*, and we shall select one for prolongational analysis.[2] The final problematic repertoire that we shall address using the groundwork laid in the previous four chapters is the neo-tonal music of the early twentieth

[1]For analyses of Wolf's songs, including the two treated here, and a theoretical discussion of the features of Wolf's musical style, see Stein 1985. For more a extensive discussion of Wolf and the scholarship that treats his life and music, see Jefferis 2004.

[2]Mitchell (1967) provides a convincing prolongational analysis of the entire *Tristan* Prelude. This prelude has historically been a proving ground for the capability of tonal theories in interpreting chromatic harmony, or, in the words of Wason (1985, 90), "the touchstone for any system of harmony aspiring to legitimacy". To follow the extensive historical bibliography of theoretical treatment of Wagner's chromatic harmony in *Tristan*, see Hyer 1989. Harrison (1994, 153–7) and Smith (1986) both offer a functional analysis of this passage, and Rothgeb (1995), Forte (1995), and Rothstein (1995) discuss the tonal origins of the enigmatic "Tristan chord". Lewin (1996) discusses the symmetry of the prelude's transformations in twelve-tone equal temperament, and Douthett and Steinbach (1998) build upon this analysis. Bailey (1985) proposes a "double-tonic complex" to explain the tonal structure of the entire opera, and Hyer (1989) uses neo-Riemannian theories (including the *Tonnetz*) to offer a renewed tonal view of the opera. My own analysis in this chapter

century. In order to demonstrate the prolongational reading of a work with non-functional passages, we shall create a prolongational network for the first of Ravel's *Valses Nobles et Sentimentales*. Finally, the parsimonious voice leading in "Ondine" from Ravel's *Gaspard de la Nuit* will allow for a prolongational reading of that primarily non-functional and non-tertian work.[3]

5.1 Enharmonic Progressions in Wolf's "Und steht Ihr früh"

Through analysis of the song, "Und steht Ihr früh am Morgen auf vom Bette" from the *Italianisches Liederbuch*, we shall examine how mentally differentiating enharmonically equivalent chords or pitches can add a layer of meaning to the interaction of music and text. In Section 3.1 we explored the possibility of an inherent phenomenological confusion in enharmonic progressions. Specifically, at the end of such a passage it may initially be unclear whether the music has moved to another tonal region, or whether it has returned to where it started. This perceptual ambiguity between moving away from and returning to tonic unlocks certain interpretive aspects of this song; it represents a "hermeneutic window" where a dramatic subtext can enter into the musical interpretation.[4] Distinguishing between C and B♯ may seem only to be an intellectual exercise. Our investigation of the enharmonicism in this music, however, not only offers interpretive insights, but also suggests that enharmonicism, by thwarting the diatonic background, presents fundamental structural challenges to common-practice tonality beyond ordinary chromaticism. The spelling rules at the core of our theory and the mod-7 networks that graphically interpret them allow us to make this distinction clear.

Figure 5.1 presents the score to this song in which Wolf sets Paul Heyse's translation of an Italian text. Figure 5.2 provides an analysis of the harmonic structure as encapsulated in a transformational network, and an English translation of the text appears in Figure 5.3. In Figure 5.2, the transformation between levels from the E-major tonic chord at level one to the initial tonic chord in level two is by (0,0). The ordered pair (0,0) between the levels, then, indicates that no transposition operation has taken place. The surface-level E chord ascends by three major thirds to the E chord that completes the cycle. To verify that these three transformations return to an E-major chord, we add the first numbers in the ordered pairs. Four (from E to A♭) plus four (from A♭ to C) plus four (from C to E) is twelve half steps.

is designed not to generate a new perspective on the *Tristan* Prelude, but instead only to resonate with and support certain aspects of many of these wonderful approaches to this extraordinary work.

[3]For a prolongational view of many of Ravel's works, and an examination of earlier tonal approaches to Ravel's music, see Chong 2002.

[4]Kramer (1990) first coined the term "hermeneutic window".

As in any twelve-tone transposition operation, the operation is addition modulo twelve. The result of our addition, twelve, thus reduces to zero because of octave equivalence. Hence we know that, at least in terms of twelve-tone equal temperament, both the first and fourth chords are equivalent E-major triads.

If there is no enharmonic shift involved, we can also expect all of the second numbers in the ordered pairs to add up to zero, when reduced modulo seven. Here, the numbers two, two, and two add only to six. This tells us that we do not finish on the same diatonic scale step as where we began. In this mod-seven representation of scale-step intervals the transposition operation T_6 is equivalent to T_{-1} (just like in the familiar mod-twelve universe T_{11} is equivalent to T_{-1}). We are now one diatonic step lower than the initial E chord, and thus theoretically on D♯♯ instead of E. The arrow that returns from this second "E" back to the E on level one thus cannot be (0,0). We must transpose up one diatonic step (by a diminished second) to return to the E-major triad on level one. This explains the transformation by zero half steps and one scale step accompanying that arrow. The low-numbered levels in mod-7 prolongational graphs, then, show how, at some degree of abstraction, certain chords that are spelled differently at the musical surface are equivalent at this more abstract level.

This process of reconciling surface-level transformations with the large-level tonic chord will therefore generate all arrows in these graphs that point from a higher-numbered level back to a lower-numbered level. When chords on the two levels are diatonically different, they will be indicated by a transposition with zero as the first number and one or six as the second number. An ordered pair of (0,1) indicates that the progression has drifted down by a diatonic step, and must be shifted back up to be completely equivalent to the starting pitch level. An ordered pair of (0,6) indicates that the progression has drifted up by a diatonic step, and must be shifted back down to be completely equivalent to the starting pitch level. In the present case, this song only drifts downward diatonically, and therefore (0,6) will not be seen in the transformational graph.

The text of the song concerns the aspects of a woman's daily activities that underscore her beauty, and how her beauty seemingly affects or enhances aspects of the world around her as well. The final four lines of the poem focus on how God has blessed the woman with her beauty, and thus has accomplished great works through her beauty. The immediacy of the simple texture, dominated by arpeggiated major chords and pedal points, underscores the primary focus of the poem's description: beauty, as found in both the woman and nature. The repeated progression by ascending major third mirrors the positive transformations that the woman's beauty effects upon the morning: clearing the skies, bringing the sun into the heavens, drawing angels to her, and brightening the lamps at church.

I would be satisfied to leave my interpretation there, as indeed the simplicity and directness of both the music and text are, in my mind, the source of their beauty. The fact that the major third cycle results in diatonic drift, however, can further enhance one's understanding of the poem's structure. Although the poem is a single stanza, Wolf's musical setting clearly organizes the text into three sections. The first involves rising in the morning, the second moves the setting to the church, and the third serves to summarize the text, providing a reason for the miraculous effects that the woman's beauty has on her surroundings. Whereas the first two sections convey a storyline, describing transformations in nature, the third section is more static, having arrived at a dramatic plateau built on the woman's beauty.

This structure is contradicted somewhat by the harmonic activity, where the only point at which Wolf breaks free from pedal point technique is at the end of the most active parts of the text. There is a dramatic increase in harmonic rhythm that begins in line 9 of the poem at m. 20 and concludes with the authentic cadence in m. 30 that starts line 14. Perhaps this "harmonic crescendo" in the music signifies a welling up of emotion on the part of the narrator resulting from the beauty he is describing.

Although the static quality of the music accompanying line 14 is virtually the same as in lines 1 and 6 (mm. 1 and 14), both the authentic cadence that initiates line 14 and other more complicated factors create a sense that this part of the music is more static than in lines 1 or 6, thus matching the textual stasis. In this regard, the transformational graph in Figure 5.2 clearly shows how diatonic spelling enhances this musical stasis. Not only does the downward diatonic drift stop around the same time the harmonic activity increases, but more importantly it seems to "settle down" to the final diatonic level where it remains during the entirety of the more static third part of the song. This "settling in," in my view, comes from an important musical contrast. The sudden shifts by ascending major third in the first section cause unexpected downward diatonic slips, whereas the more functional progression connecting C (chord 6) and E (chord 16) at the end of the second section enables a smoother transition into the lowest diatonic position of the tonic E.

Figure 5.4 translates the transformational network in Figure 5.2 into Schenkerian prolongational notation. While the prolongational sketch in Figure 5.4 does not easily show both the change in diatonic spelling and the prolongation through the enharmonic progressions, the transformational network in Figure 5.2 helps to clarify the relationship beween the diatonic spelling and the prolongational view of the piece.[5] Specifically, the arrows pointing from the E-major chords on level two of the graph back up to the E-major chord on

[5]The prolongational network, however, may be seen as oversimplifying the tonal structure, simply because it cannot include features such as the interruption at the end of m. 29. We shall return to this matter in Section 5.4.

level one involve enharmonic shifts. This does not necessarily negate the possibility of prolongation, but rather suggests that it may be a different kind of prolongation than the kind normally seen in tonal music. The transformational network format thus allows for a clear perspective on enharmonic progressions from both transformational foreground and tonal background viewpoints. We shall continue to weigh the relative merits of prolongational and transformational analysis techniques in the next section, where we shall examine how Wolf also uses directional tonality to transcend traditional harmonic practice.

Figure 5.1: Wolf, “Und steht Ihr früh am Morgen auf vom Bette”, Score

Figure 5.1, continued

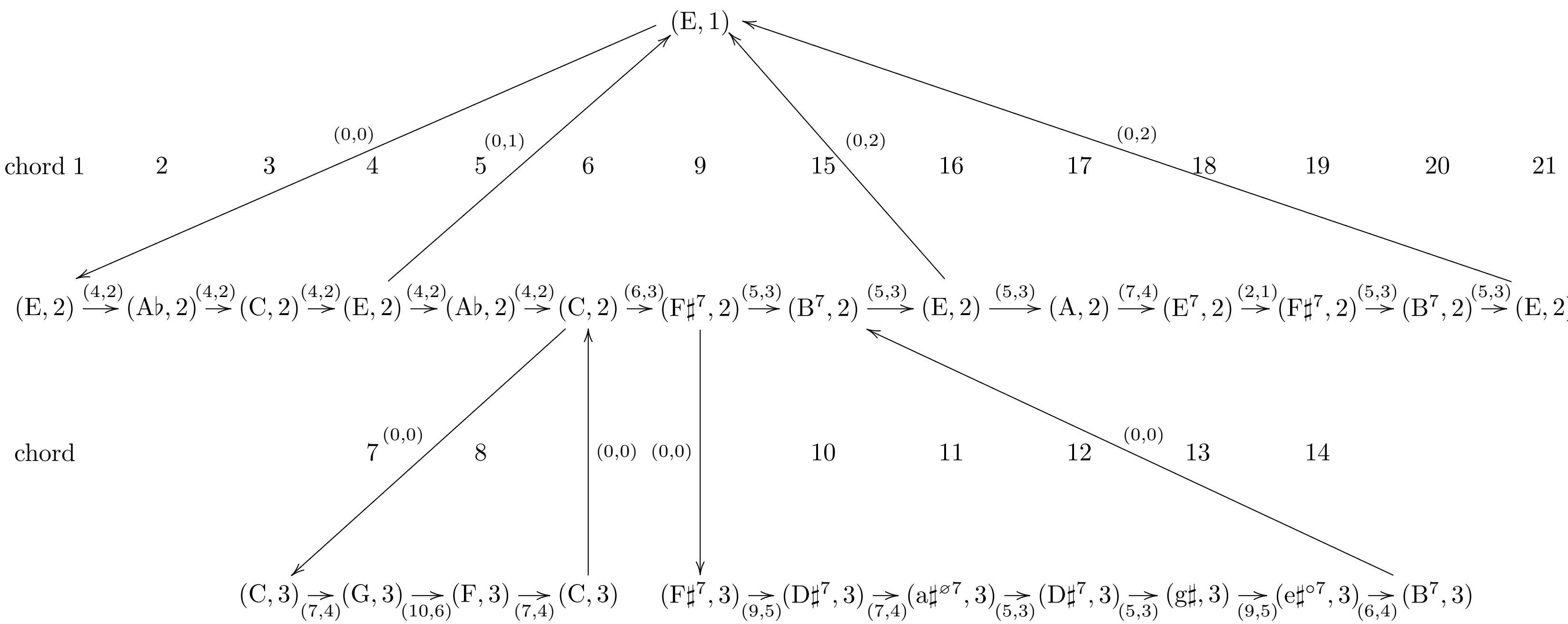


Figure 5.2: Transformational network describing Wolf, "Und steht Ihr früh"

Und steht Ihr früh am Morgen auf vom Bette,	And when you rise early from your bed,
Scheucht Ihr vom Himmel alle Wolken fort,	You banish every cloud from the sky,
Die Sonne lockt Ihr auf die Berge dort,	You lure the sun onto those hills,
Und Engelein erscheinen um die Wette	And angels compete to
Und bringen Schuh und Kleider Euch sofort.	Bring your shoes and clothes.
Dann, wenn Ihr ausgeht in die heil'ge Mette,	Then, when you go out to Holy Mass,
So zieht Ihr alle Menschen mit Euch fort,	You draw everyone along with you,
Und wenn Ihr naht der benedeiten Stätte,	And when you near the blessed place,
So zündet Euer Blick die Lampen an.	Your gaze lights up the lamps.
Weihwasser nehmt Ihr, macht des Kreuzes Zeichen	You take holy water, make the sign of the cross
Und netzet Eure weiße Stirn sodann	And moisten your white brow,
Und neiget Euch und beugt die Knie ingleichen—	And you bow and bend the knee—
O wie holdselig steht Euch alles an!	Oh, how beautifully it all becomes you!
Wie hold und selig hat Euch Gott begabt,	How sweetly, blessedly has God endowed you,
Die Ihr der Schönheit Kron empfangen habt!	Who have received the crown of beauty.
Wie hold und selig wandelt Ihr im Leben;	How sweetly, blessedly you walk through life;
Der Schönheit Palme ward an Euch gegeben.	The palm of beauty was bestowed on you.

Figure 5.3: English translation of Heyse, "Und steht Ihr früh am Morgen auf vom Bette"

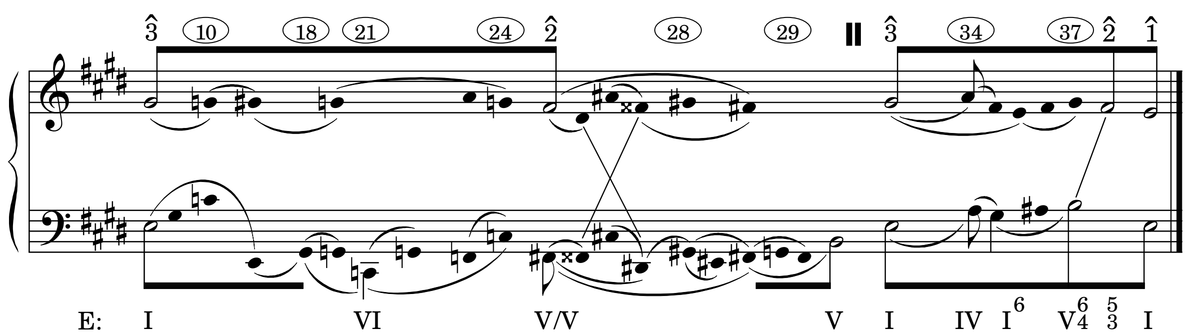


Figure 5.4: Prolongational Sketch of Wolf, "Und steht Ihr früh" based on Figure 5.2

5.2 Directional Tonality in Wolf's "Der Mond"

Analysis of the song "Und steht Ihr früh" in the previous section demonstrated how diatonic transformations might clarify prolongations that involve enharmonic progressions. A similar transformational approach is also useful for the analysis of other extraordinary features of nineteenth-century tonal practice.[6] Figure 5.5 shows the score to Hugo Wolf's "Der Mond" from the *Italianisches Liederbuch*, Figure 5.6 presents a mod-7 transformation network describing the song's prolongational structure, and Figure 5.7 gives an English translation of the text. Figure 5.8 gives a middleground sketch derived from the prolongational network in Figure 5.6.[7] My sketch in Figure 5.8 does not account for any of the phenomenological confusion that may result from the song's ending a third below where it began. Such a tonal reading nevertheless offers a strategy for listening to the piece in order to make sense of the relationship between the beginning and the end. Knowing how the song ends, one may be able to hear the opening contextually with reference to the ending key. In Figure 5.8, the mediant chord serves as a harmonic substitute for the opening tonic. Since there is no internal musical evidence of this substitution at the opening of the piece, it is perhaps more accurate to conceive of the piece as missing its opening. It is as though the listener has arrived late and enters the recital in the middle of the first song. This interpretation of the directional tonality seems to resonate with the effect of hearing the opening of the text. The poem's first few lines seem like such a *non sequitur* in the context of the song cycle that one may wonder about the significance of this story about the moon complaining.

All of the chords that are shown with open noteheads in Figure 5.8 appear on level one of Figure 5.6, though Figure 5.8 adds the repetition of the final V–I progression that supports the *Urlinie* descent from $\hat{3}$ to $\hat{1}$. All chords that have stems in Figure 5.8, except for one extra D♭ chord, are present on level two of Figure 5.6. (The bass-clef slur from B♭ in measure 6 to G♭ in measure 9 indicates that the D♭ in measure 8 is a more foreground event.) In an ideal transformational graph of this type, the deepest level could contain only one chord: the tonic. In this case, however, the song's directional tonality eliminates this possibility. If one were to imply a conceptual tonic chord preceding the opening E♭-minor chord, a closed tonal graph would obtain. Another possibility (one that I find more compelling) would be to show an arrow from the single tonic chord on level 1 to the first E♭-minor chord on level 2 that displays a transformation other than (0,0). In this case, the ordered pair accompanying

[6]For background on tonal pairing and directional tonality, and other useful analytical approaches to music that displays several different types of multiple-key combinations, see Kinderman and Krebs 1996.

[7]Many interesting foreground details of the song—especially in the vocal line—are missing in this middleground sketch, which is designed simply to show the value of the auxiliary-cadence interpretation of the song's directional tonality.

that arrow (from C♭ major to E♭ minor) would be (4,2). Whereas the Schenkerian sketch essentially normalizes the song with regard to its tonal interpretation, the transformational graph highlights the tonal problem created by directional tonality. In this case, the analysis dramatizes one of the central issues concerning the song's tonal structure: If it is going to end in C♭ major, why does it begin in E♭ minor?

How this tonicized E♭-minor chord functions with respect to the tonal scheme is revealed in the tonal relationships that unfold over the course of the short song. The way in which the song clarifies the function of the first chord parallels the structure of text, which begins with a personification of the moon (in E♭ minor), then continues by blaming the narrator's lover for the moon's distress (in G♭ major), then mourns the loss of two stars from the heavens (in G♭ minor), and finally contextualizes the entire poem by revealing the metaphor of the stars as the woman's eyes (in C♭ major). The listener thus only discovers the true key of the piece when the real reason for the moon's complaint and the missing stars is disclosed.

Composers of the nineteenth century capitalize on the ambiguities of chromatic harmony in many different ways. In the present case, Wolf withholds the actual key of the piece until the end, just as the text withholds the key to the central metaphor of the poem until the end. In the previous example (Figure 5.1), enharmonic equivalence created a confusion between different scale degrees that normally remain autonomous in tonal music. In both cases, the transformational graphs highlight the ambiguities while the Schenkerian sketches downplay them. The strength of the Schenkerian approach is that it shows a way out of the tonal ambiguity and offers the cognitive tools for attaining a tonal hearing of the piece. When the prolongational structure is reconceived from a transformational viewpoint, the ambiguities of chromatic harmony become apparent because they create inconsistencies in the graph structure. While the diatonic transformational viewpoint is also tonally normative, it carries less theoretical baggage: No chord has any expectation of moving to any other chord in particular; there is no requirement for a monotonal analysis; there is no need to reconcile the music to an *Ursatz* structure; and post-tonal works can receive similar types of analysis. I would not wish to abandon Schenkerian analysis, even in the face of highly chromatic music. It remains the more useful interpretive tool for tonal music. Nevertheless, the mod-7 transformational viewpoint can serve to elucidate the ambiguities of chromatic harmony. In the next section, we shall continue to use a diatonic approach in order to analyze the highly chromatic opening of Wagner's *Tristan* Prelude using both prolongational and transformational methodologies.

Figure 5.5: Wolf, "Der Mond hat eine schwere Klag' erhoben" (1890), Score

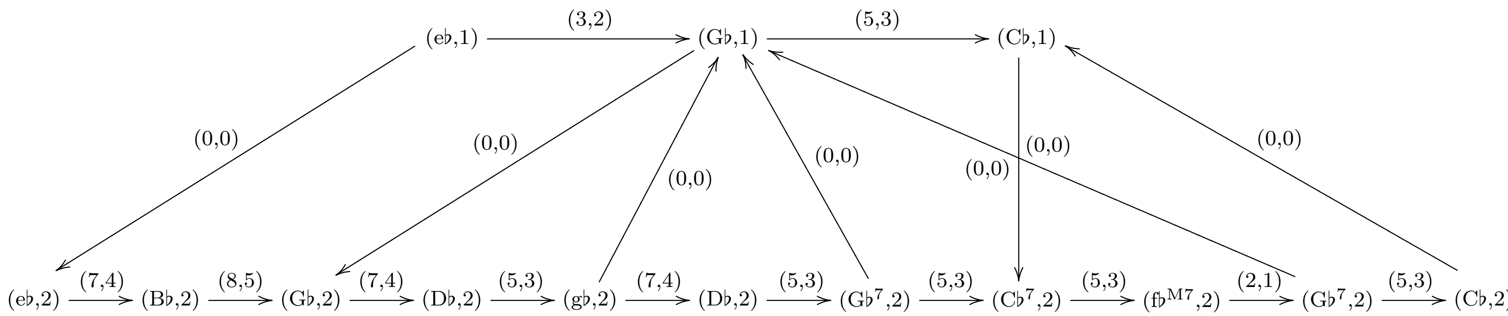


Figure 5.6: Transformational graph of Wolf, “Der Mond hat eine schwere Klag’ erhoben”

Der Mond hat eine schwere Klag' erhoben	The moon has raised a grave complaint
Und vor dem Herrn die Sache kund gemacht;	And made the matter known unto the Lord:
Er wolle nicht mehr stehn am Himmel droben,	He no longer wants to stay in the heavens,
Du habest ihn um seinen Glanz gebracht.	For you have robbed him of his radiance.
Als er zuletzt das Sternenheer gezählt,	When he last counted the multitude of stars,
Da hab es an der vollen Zahl gefehlt;	Their full number was not complete;
Zwei von den schönsten habest du entwendet:	Two of the fairest you have stolen:
Die beiden Augen dort, die mich verblendet.	Those two eyes that have dazzled me.

Figure 5.7: English Translation of Heyse, "Der Mond hat eine schwere Klag' erhoben"

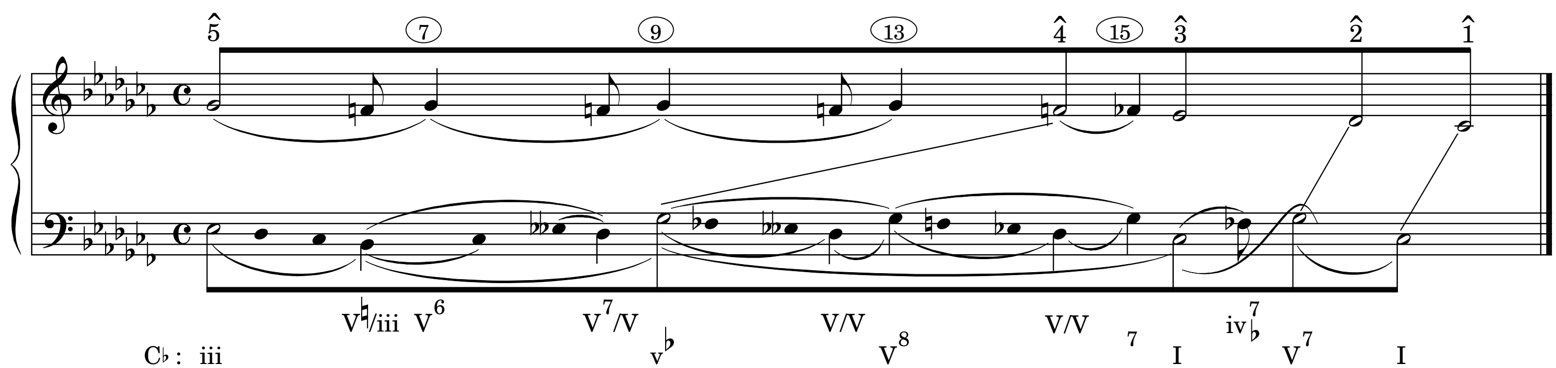


Figure 5.8: Prolongational Sketch of Wolf, "Der Mond" based on Figure 5.6

5.3 Post-Functional Progressions in Wagner's *Tristan und Isolde*

As we have seen in the previous two sections, mod-7 transformations are useful for clarifying enharmonic progressions and directional tonality. They are also useful for interpreting extreme examples of nineteenth-century chromatic harmony from a diatonic tonal perspective. While examining the affinities of many examples of chromatic harmony to transformations within octatonic and hexatonic collections is often useful, it is also a worthwhile pursuit to place the chromatic passages analytically within the work's overall diatonic context. For this purpose we shall find the guidelines in Table 2.2 to be useful. Providing a diatonic context for highly chromatic passages may also serve to clarify functional ambiguities that tend to obscure the possibility of prolongational analysis. The unusual chord resolutions present at the opening of the Prelude to Wagner's *Tristan und Isolde* will provide a preliminary example of chord successions involving exclusively atypical resolutions of tertian chords that normally resolve functionally. A piano reduction of the excerpt is given in Figure 5.9(a). Typical functional resolutions begin with the B dominant-seventh chord (6) resolving to the E dominant ninth chord (7).

It is fortunate that there are many dominant-seventh chords in the passage, as they can be used to decide the key to use in deciding the diatonic spelling. The resulting key scheme is A in mm. 0–5, C in mm. 6–9, E in mm. 9–15, and A in mm. 16–17. There are two viable approaches to the diatonic spelling of this excerpt. The distinction between these two approaches rests on the ambiguity in Rules 1 and 2 of Table 2.2 regarding when to prefer tertian chord spellings (functional chords) over scale-based chord spellings (altered dominants). Figures 5.9(a) and 5.9(b) show these two interpretations. Figure 5.9(a) strictly follows the scale degrees in Table 2.1 for the keys in the key scheme given by the dominant-seventh chords. In Figure 5.9(b) every verticality is spelled as a tertian chord and participates in diatonic-interval root motion following the guidelines in Table 2.2.[8] The difference between the two interpretations is a matter of dissonance. Whereas the first interpretation presents passing chords as dissonant entities, the second interpreatation prefers all consonant chords (which the ear may accept more readily at the typical performance tempo).[9]

The scale-based spelling has distinct advantages in terms of defining the function of each pitch and verticality (and tuning the chords accordingly). A linear sketch of this spelling of

[8]Note the non-diatonic root motion between chord 2 (E^7) and chord 3 ($a\flat^{\o 7}$). This is the result of a change of tonic (and thus diatonic scale) between the two chords. Also, the apparent non-diatonic root motion from the $A\flat^{+6}$ at the end of m. 10 to the $B^{7}_{\flat 5}$ and B^7 in m. 11 is also functionally mitigated by the adherence to the guidelines in Table 2.2. Further, the acoustical method for finding root representatives outlined in Table 4.1 revises our root progression A♭, B, and B to the root progression C, F, and B, which reveals a better diatonic basis for the progression than our original observation suggested.

[9]Smith (1986) also capitalizes on the multiple interpretations of this passage resulting from opposing linear and harmonic views.

the Prelude, while abandoning the grounding of our theoretical apparatus, would highlight the functional discharge of chromatically inflected scale degrees. Following each chromatic line in this way forms a rewarding way of hearing the music. With regard to the use of mod-7 transformations, however, it will be more useful for us to be able to refer to the chords by their root and quality. Figure 5.10 uses typical designations for the tertian chords in Figure 5.9(b) to provide a mod-7 prolongational network describing the passage. Level 1 of the graph shows the prolongation of the E dominant seventh chord between chords 2 and 7, and the deceptive resolution of that chord in the key of A minor. This prolongational view follows most other scholars' tonal views of the passage. Level 2 includes the intermediate dominant sevenths prolonging the E^7 by arpeggiation and the neighboring "Tristan" chords that precede each.[10] Finally, level 3 shows all of the intervening chromatic passing chords between each half-diminished and dominant seventh pair. The prolongational structure shown in the network is translated into a prolongational sketch in Figure 5.11. The linear analysis displays specifics of the voice leading that may only be implied in the transformational network and certainly offers more sophistication and analytical detail. But without the formalism of the mod-7 graph and the adherence to tonally based diatonic spelling strictures, the analytical decisions leading to the unique details of the sketch in Figure 5.11 may seem to have been arbitrary.

Our two diatonic readings of this familiar excerpt suggest that any example of chromatic harmony may be analyzed diatonically provided that the analyst be able to determine the keys through which the music moves. A consistent diatonic reading of other even more highly chromatic music may thus be possible using the spelling strictures in Table 2.2, as long as the key can be determined from the diatonic collection, key-defining sonorities, or traditional functional resolutions of dissonances.[11] The value of using a diatonic model for such highly chromatic music lies in the diatonic scale's ability to imbue the chromatic collection with tonal function, and thus participate in a tonal reading of the entire piece. In the next section of this dissertation, we shall take a similar approach to an example where functional progressions still define the key, but the complexity of the chords used obfuscates the root analysis and sometimes also the chords' functions.

[10]It would certainly also be correct to show the half-diminished neighbor chords on a shallower analytical level than the dominant sevenths. This has not been done for the sake of clarity and conciseness in the graph.

[11]Our analytical apparatus may thus form an interesting (though not definitive) test of Schoenberg's claim that his "atonal" music is in fact "pantonal", implying many keys.

(a) Scale-Based Spelling

Figure 5.9: Wagner, *Tristan und Isolde*, Prelude, mm. 1–17, Score

(b) Tertian Spelling

Figure 5.9, continued

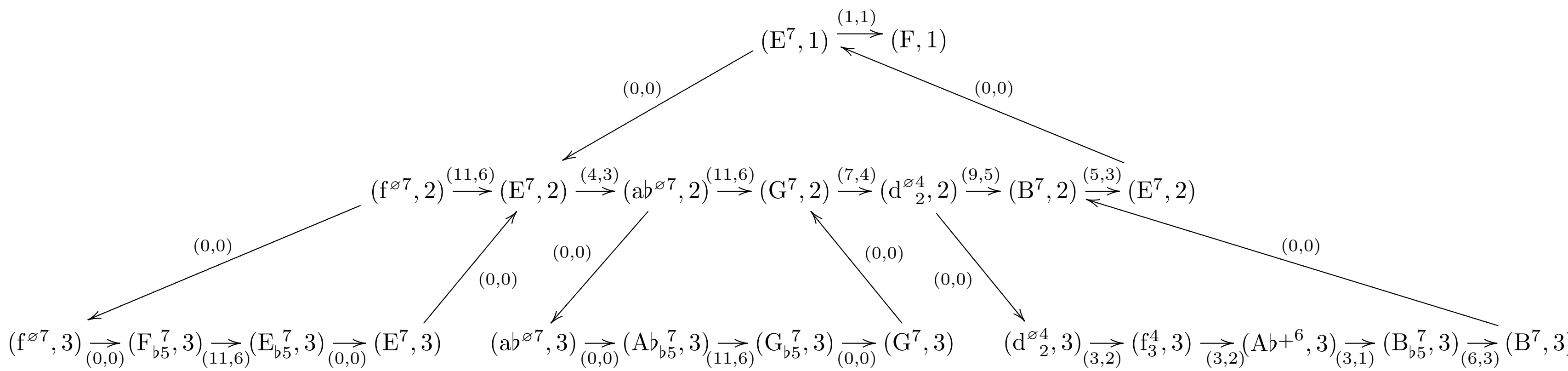


Figure 5.10: Transformational Graph based on Figure 5.9(b) (Tertian Spelling)

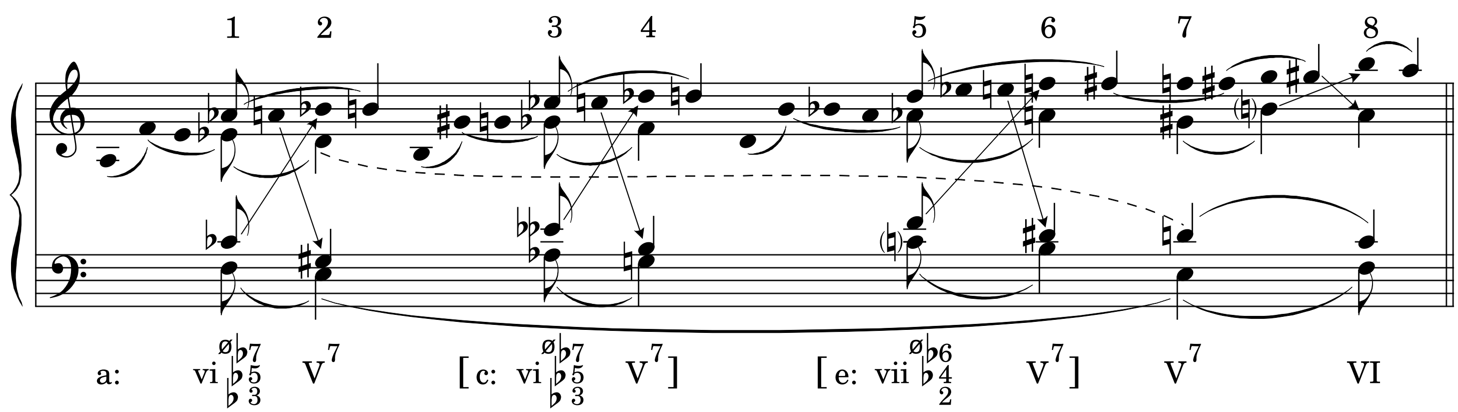


Figure 5.11: Prolongational Sketch based on Figure 5.10

5.4 Non-Tertian Progressions in Ravel's *Valses Nobles*

Now that we have seen how a diatonic approach can clarify tonal function in highly chromatic passages, we shall now turn to an example where our root-finding method will aid in clarifying the tonal function of non-tertian and extended-tertian chords. We shall only explore the first waltz in Ravel's *Valses Nobles et Sentimentales*, but an analysis the rest of the suite would reveal a largely consistent extended-tonal language throughout.[12] The reductive analysis of Ravel's music is hardly novel or inappropriate since, as reproduced by Chong (2002, Vol. 2, p. 1, Ex. A.1), Ravel himself reduced one of the *Valses Nobles et Sentimentales* (VII) down to a basic diatonic functional progression. This is largely the approach taken by Chong in his own analyses of Ravel's other works, reconciling extended-tonal features of each work to typical Schenkerian diminution patterns and functional progressions. The analytical techniques that we have developed in this dissertation simply codify and theoretically support this technique of simplifying unusual tonal features down to their basis in common-practice tonality. My use of root (representative) analysis for Ravel's "polychords" is supported by Kaminsky (2004). I do not claim that the particular fundamental-bass analysis provided here is in any way definitive.

Figure 5.12 displays the score to the first of Ravel's *Valses Nobles et Sentimentales* in G major. Diatonic spelling in Figure 5.12 is based on the strictures in Table 2.2. Recall that the only information required for diatonic spelling using Table 2.2 is the current key center and mod-12 pitch class numbers of all notes. The reference scale for mm. 1–8 is based on the tonic pitch class G. Then mm. 9–14 expand an extended-tertian applied dominant to A. The extended A chord that arrives in m. 15 tonicizes D and is arpeggiated through m. 19 before resolving to a D chord in m. 20. While a D pedal point extends through m. 30, the diatonic spelling of the chords above the pedal point has been determined independently of the repeated low D. An extended-tertian but functional V/V–V–I progression in E that resolves on the downbeat of m. 33 suggests a diatonic interpretation based on that key in mm. 33–38. Beginning in m. 39, the music becomes more harmonically active and ambiguous, culminating in a complete chromatic circle-of-fifths progression in mm. 57–60.[13] In mm. 39–44 the music alternates between extended dominant chords built on C♯ and G. The marking "enh." that I have added to m. 45 and also later in the movement indicates that an enharmonic shift that is not warranted by the spelling rules given in Table 2.2 has been made in the music for the sake of reading ease. In all such cases, there is a clear shift from flats to sharps or vice versa,

[12] McCrae (1974) provides colorful analyses of each of the waltzes, and also discusses their possible relationships with earlier sets of waltzes by Schubert (his own *Valses Nobles*, D. 969, and *Valses Sentimentales*, D. 779) and Schumann (*Papillons*, Op. 2).

[13] For a mathematical model of the structure of this interesting sequence, see (Smith 1975).

and the maintenance of a single diatonic spelling of the common pitch class(es) between the two chords will rectify the convenience spelling to the rule-derived spelling. A majority of the chords between mm. 45 and 60 are extended dominants, and the diatonic spelling of this section is governed by tertian spelling and the maintenance of common tones. An altered reprise of the introductory four measures (cf. mm. 1–4) appears in mm. 61–64, tonicizing G. The introduction of F♮ in mm. 67–70 suggests a tonicization of C. No C chord appears in m. 71, however, which tonicizes A in a manner similar to mm. 11–14. The arpeggiation of the extended-tertian A-dominant chord in mm. 75–78, in this case, resolves one measure early to D in m. 79 as the dominant to G (cf. mm. 15–20).

Figure 5.13 translates our extended-tertian analysis into a mod-7 prolongational network. While my prolongational reading here is largely based on Schenkerian paradigms, the deepest level of the transformational network is different from the background of the deep middleground sketch shown in Figure 5.14. This is because there is no notation in prolongational networks for interruption.[14] For this reason, Figure 5.14 gives a more comprehensive tonal reading of the piece. The traditional 3-line *Ursatz* with an interruption that is shown in the sketch suggests a relationship between the form of this waltz and other traditional dance forms.[15] The highly functional background may also help us make sense of the complex non-functional foreground progressions and non-tertian chords. In the next section, we shall use mod-7 transformations to posit a prolongational reading of an even more extravagantly non-functional movement by Ravel.

[14]We could certainly invent one, but it would be extraneous to the group structure of the mod-7 transformation networks. For example, we could reverse the direction of the arrow from the tonic to the first structural dominant to show it as "back-relating".

[15]My preference would be to call this a rounded binary form, but it certainly has ternary characteristics as well. Both forms, of course, may be read with an interruption.

Figure 5.12: Ravel, *Valses Nobles et Sentimentales* (1911), I, Score, Spelled using Table 2.2

Figure 5.12, continued

Figure 5.12, continued

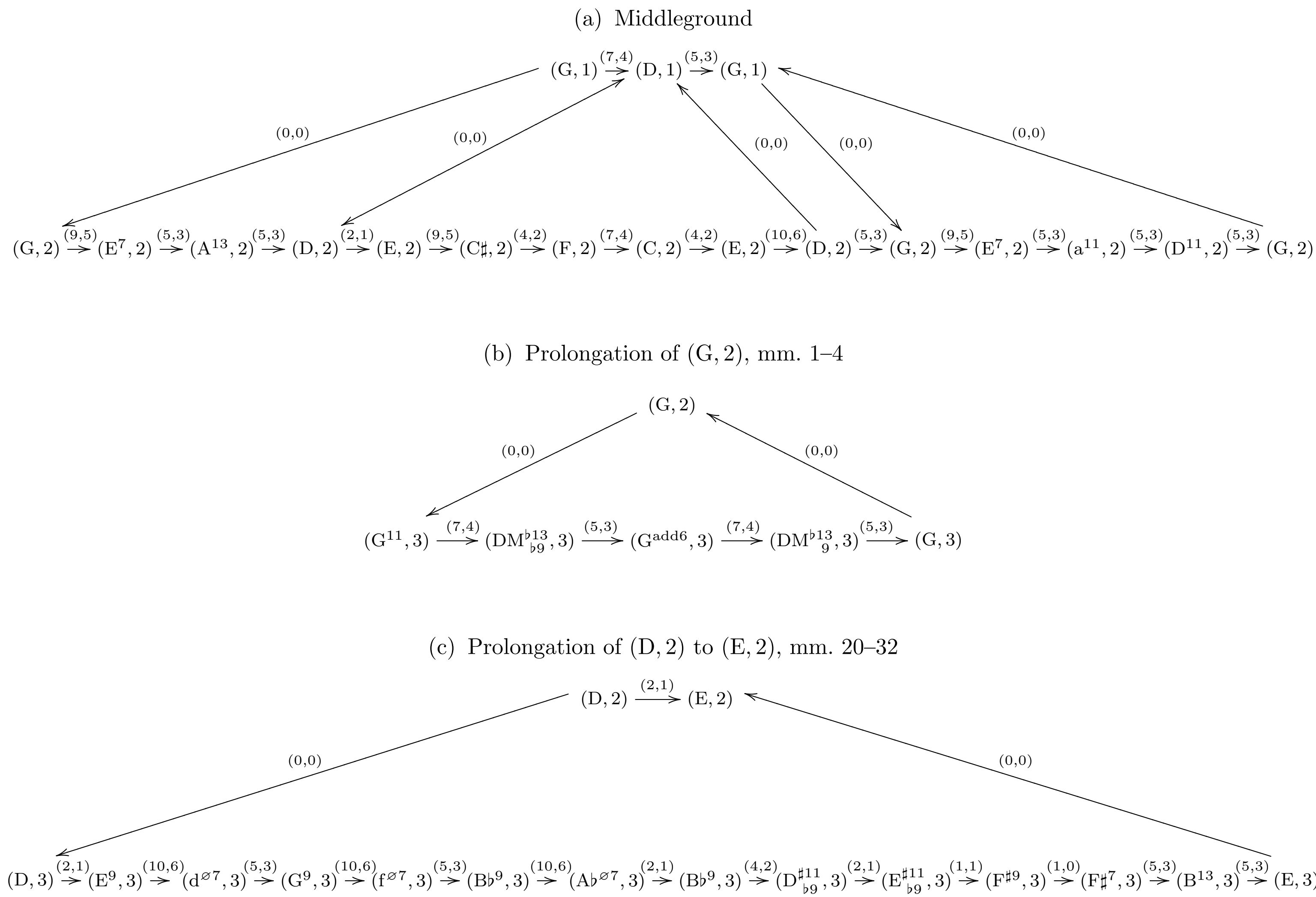
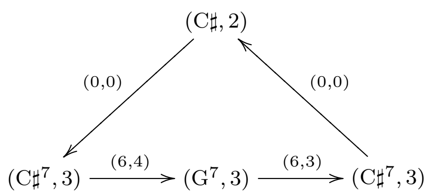
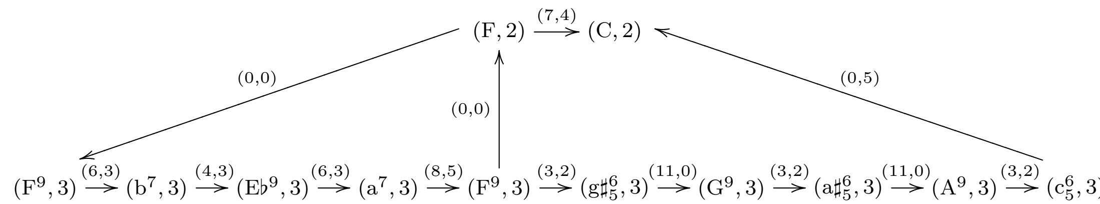


Figure 5.13: Transformational Graphs of Ravel, *Valses Nobles et Sentimentales*, I

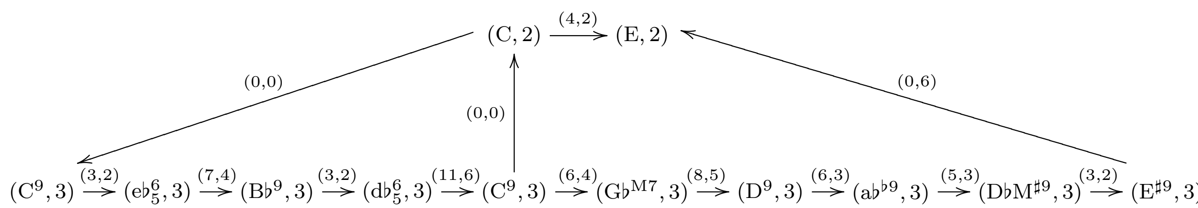
(a) Prolongation of (C♯, 2), mm. 39–44



(b) Prolongation of (F, 2) to (C, 2), mm. 45–50



(c) Prolongation of (C, 2) to (E, 2), mm. 51–55



(d) Prolongation of (E, 2) to (D, 2), mm. 55–57

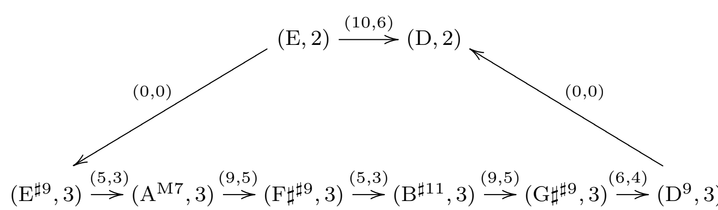
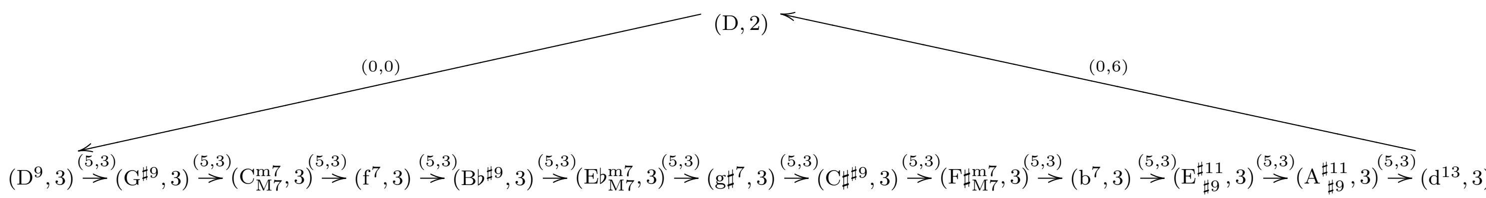


Figure 5.13, continued

(e) Prolongation of (D, 2), mm. 57–61



(f) Prolongation of (D, 2) to (G, 2), mm. 61–65

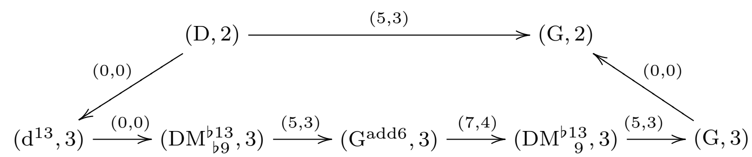


Figure 5.13, continued

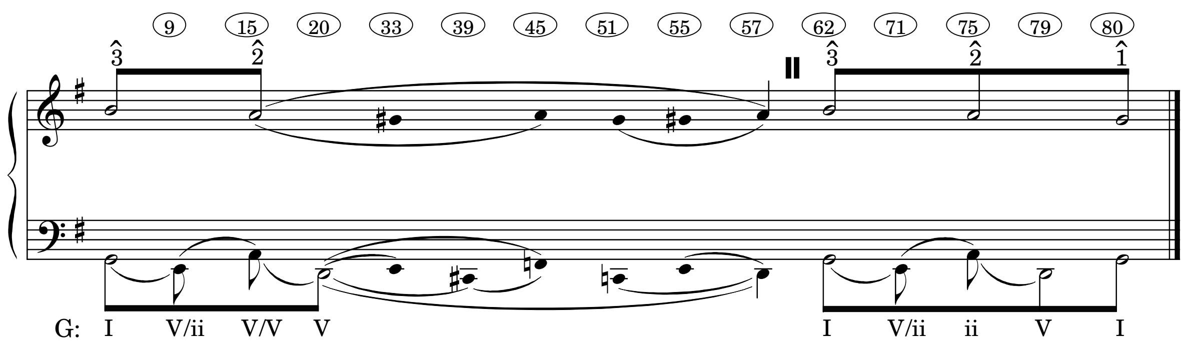


Figure 5.14: Prolongational Sketch of Ravel, *Valses Nobles et Sentimentales*, I

5.5 Post-Functional Non-Tertian Progressions in Ravel's *Gaspard de la Nuit*

The movement by Ravel that we have just examined is governed by traditional form and delineated by a functional cadential structure. Hence it will be worthwhile to attempt to use our prolongational theory to address a movement by Ravel that follows a freer form and displays more ambiguity with respect to traditional tonal harmony.[16] Figure 5.15 shows the score to "Ondine" from Ravel's *Gaspard de la Nuit*, and Figure 5.16 provides a middleground prolongational sketch of the work. We can evaluate the prolongations shown in Figure 5.16 using the set of criteria given in Table 4.4. For the most part, the sketch satisfies question 1 concerning whether the passage begins and ends on the same harmony. In the few cases where it does not, the passing chords either smooth the progression from one structural (open-notehead) chord to another, or they connect two chords that can be said to possess the same harmonic function. One such instance is the transitory progression in mm. 15–23, which forms the second part of the transformational network in Figure 5.17. The return in mm. 15–16 to the harmony, melody, and texture from m. 1 marks the beginning of this elaborated tonal motion. The goal chord is the G♯ dominant harmony in m. 23 that supports $\hat{4}$ of the *Urlinie*. Another passage that violates the desideratum of the first consideration in Table 4.4 can be found in mm. 63–67, a prolongation that connects a G♯ chord with a B-minor chord. This prolongation is graphed in Figure 5.18. In this case, the two chords can be said to serve the same harmonic function. Figure 5.16 shows that they both take part in a dominant prolongation. Specifically, the G♯ in m. 63 begins a bass arpeggiation through the B in m. 67, which is inflected to B♯ in m. 73 and finally returns to G♯ in m. 81.

Now let us address the second consideration in our prolongational rubric, regarding acoustical stability. Many of the structural chords in my sketch are transpositions of the opening major triad with an added minor sixth. Assuming that the neighbor formations in the accompaniment pattern can allow us to reduce the added sixths out as non-chord tones,[17] the structural harmonies are often more stable than the intervening chords. The first part of Figure 5.17 graphs the work's initial prolongational span (mm. 1–15), and Table 5.1 uses our instability units from Section 4.2 to show the relative acoustical stabilities of each of the chords. Recall that the instability units, where 1 is the most stable, and higher numbers

[16]Bhogal (2004) compares this piece to sonata form, provides a deep middleground bassline sketch, and then provides a gender-based hermeneutic interpretation of the piece. Haapanen (2004) also extends a Schenkerian approach to treat this work.

[17]The fourth consideration in our list of prolongational criteria may aid in supporting this reading. Bhogal (2004) and others have asserted that this neighbor motion is the primary motive of the piece. My own analysis does not focus on this interesting observation, but may certainly be enriched by this point of view.

indicate relative degrees of acoustical instability, was one interpretive measure that helps us to gauge a chord's relative salience. The first and last chord, when reduced to their triadic forms, have instability level 5, but with the added minor sixths, the instability rises to 13. Interestingly, the added fourth that appears in the F♯-minor chord in m. 11 does not change the instability of that chord at all. Many of the later prolongational spans feature a clearer distinction between acoustically simple chords and more complex extended chords. From Table 5.1 we can see that, even if the prolonging chords in this passage are indeed more stable than the structural chords, at least the structural chords are acoustically distinguished from the intervening harmonies.

Furthermore, these added sixth chords are motivically significant. Following the many extravagant excursions, the music frequently returns to an added sixth chord presented in a texture similar to the opening. The most obvious of these returns to the opening texture and chord type can be seen in mm. 31, 42, and 81. Further, these particular excerpts also present a melody that is the same as or similar to the left-hand melody in m. 3, thus enhancing this thematic connection. These motivic connections have played a significant role in my reading of prolongation in this work. My sketch shows the first chord of each reprise of the opening texture and harmony as initiating a new prolongational span at some level of the structure. In addition to mm. 1, 31, 42, and 81, these returns to the opening texture are shown as initiating prolongational spans in m. 24, m. 46 (see especially m. 48), and m. 75, where the texture finally coalesces in m. 76. The last reminder of the opening texture and harmony is the final sonority of the piece in m. 90.

A second motive in the piece also appears at the beginning of many prolongational spans. This motive is harmonically marked by the use of tritone-related neighbor chords. For example, in m. 43 the progression $D\sharp^{\text{add }6}$ A^9 $D\sharp^{\text{add }6}$ accompanies the main melody (from m. 3). This progression continues a gestural/rhythmic motive begun in m. 39 where an accompaniment pattern similar to the opening is interrupted by a quickly ascending and descending flourish that arpeggiates the neighbor chord. This gesture and chord progression are combined to accompany a new melody in mm. 46, 51, 58, etc. This new melody typically involves a stepwise scalar ascent followed by an upward leap to a descending appoggiatura. This appoggiatura often coincides with the flourish and the neighbor chord. In my sketch this motive initiates each new prolongational span within the linear intervallic pattern that stretches from m. 37 to m. 63 (i.e. mm. 42, 46, 51, 58, and 61). Thereafter it is liquidated in a much more active linear intervallic pattern from m. 63 to the climax of the piece in m. 67.

The prolongation of a B-minor chord in m. 67 inverts the stepwise-ascent motive. This inversion can be seen in the right-hand eighth notes that initiate each flurry of thirty-second notes. The appoggiatura is now gone from this melody, since the leap at the end of the stepwise line in m. 67 is now on a weak beat. To begin the dénouement, m. 67 repeats with the bass an octave higher and the melody an octave lower in m. 68. Through the rest of the piece, we continue to hear neighbor motions reminiscent of this motive, but the ascending stepwise melody never reappears.

The types of prolongation described here certainly follow traditional functional paradigms to a lesser degree than the earlier examples we have examined. Nevertheless, we were still able to use harmony and voice leading to posit a prolongational reading. In music with such a highly structured tonal scheme rooted in tertian harmony, it might be unsatisfying to provide a legitimate tonal analysis that stops short of showing prolongation. In the final chapter of this dissertation, I shall briefly summarize some of the conclusions that we have drawn in the first five chapters, and I shall discuss the possibility of using or expanding this theory to examine other repertoires.

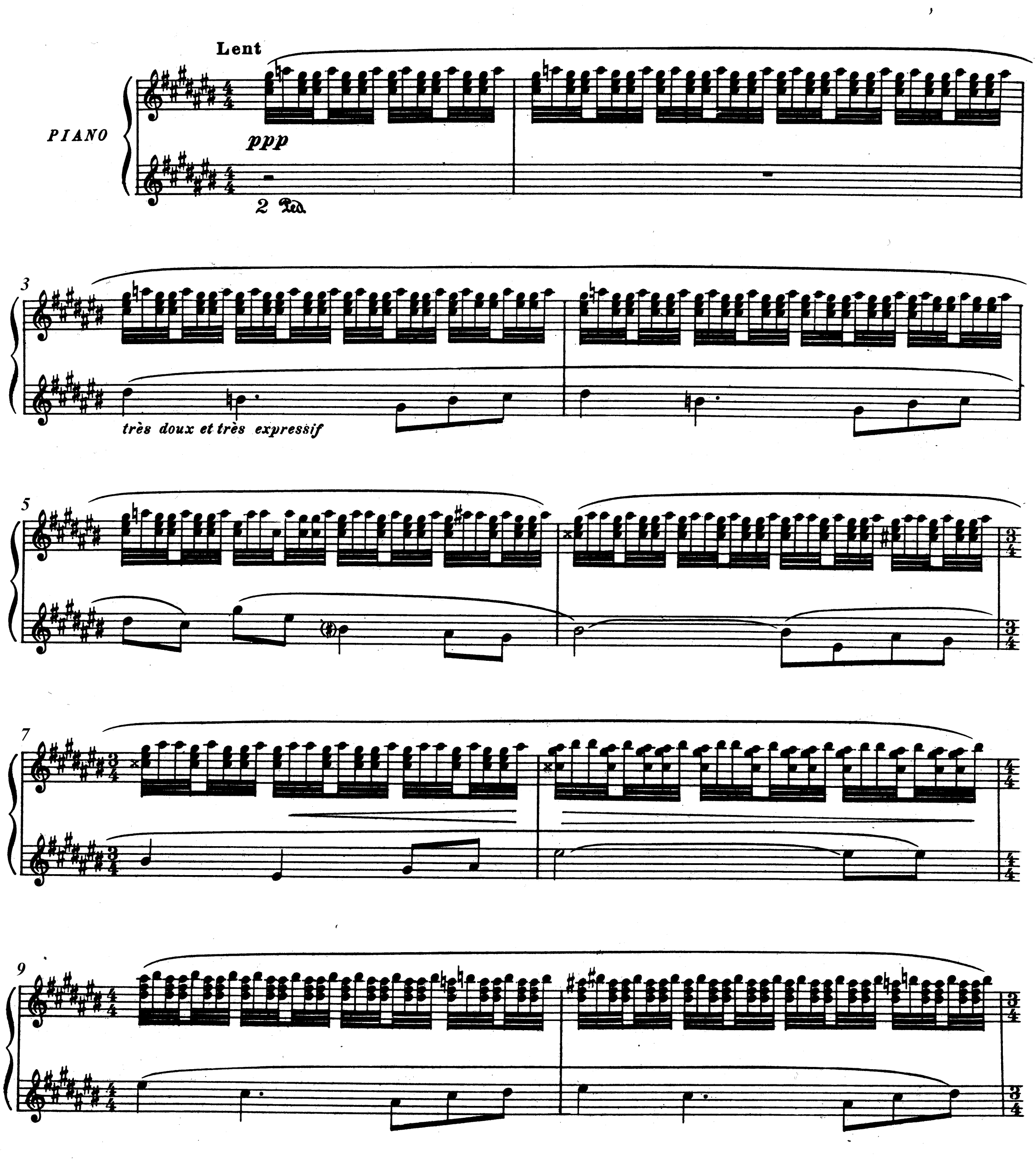


Figure 5.15: Ravel, *Gaspard de la Nuit* (1908), "Ondine", Score

Figure 5.15, continued

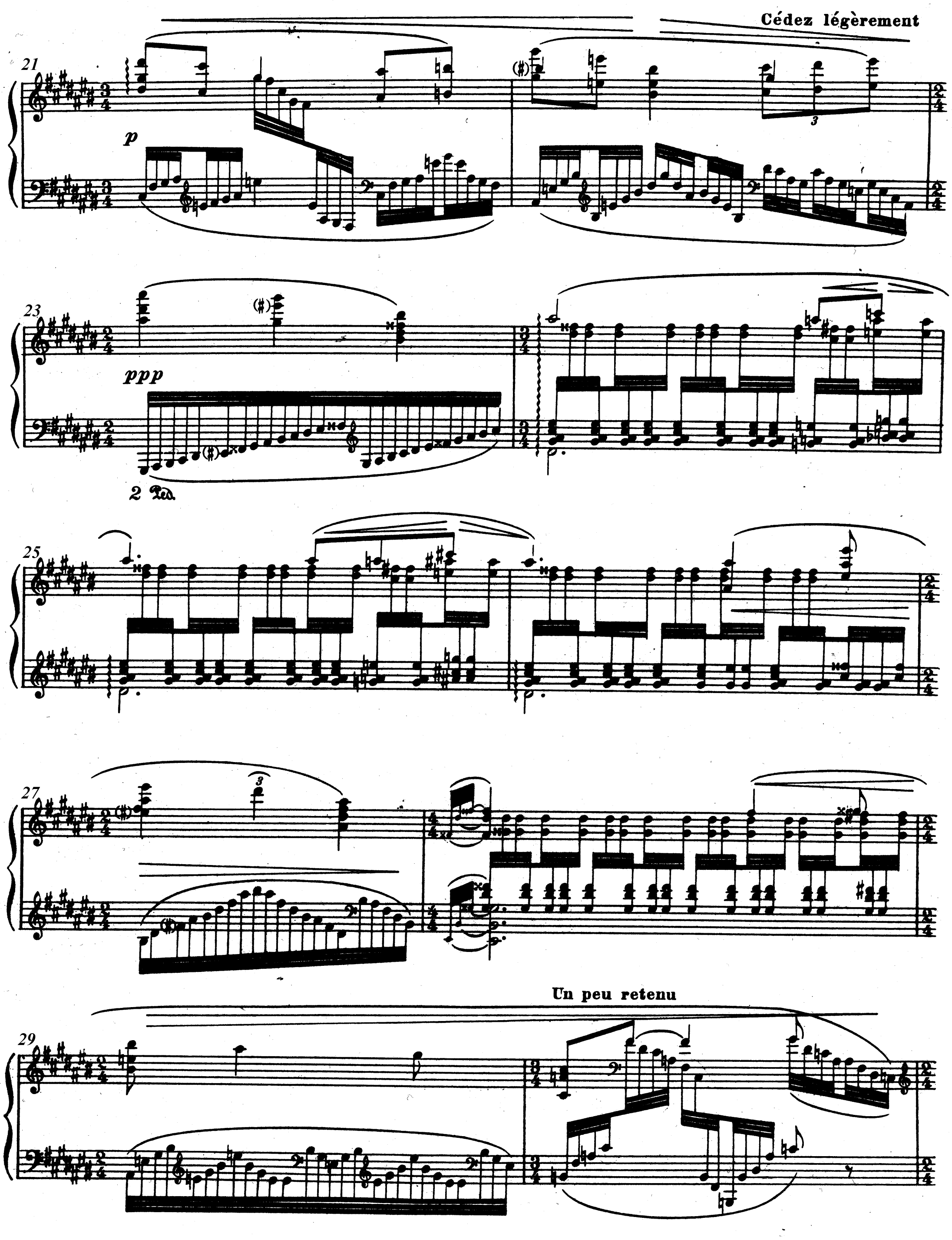


Figure 5.15, continued

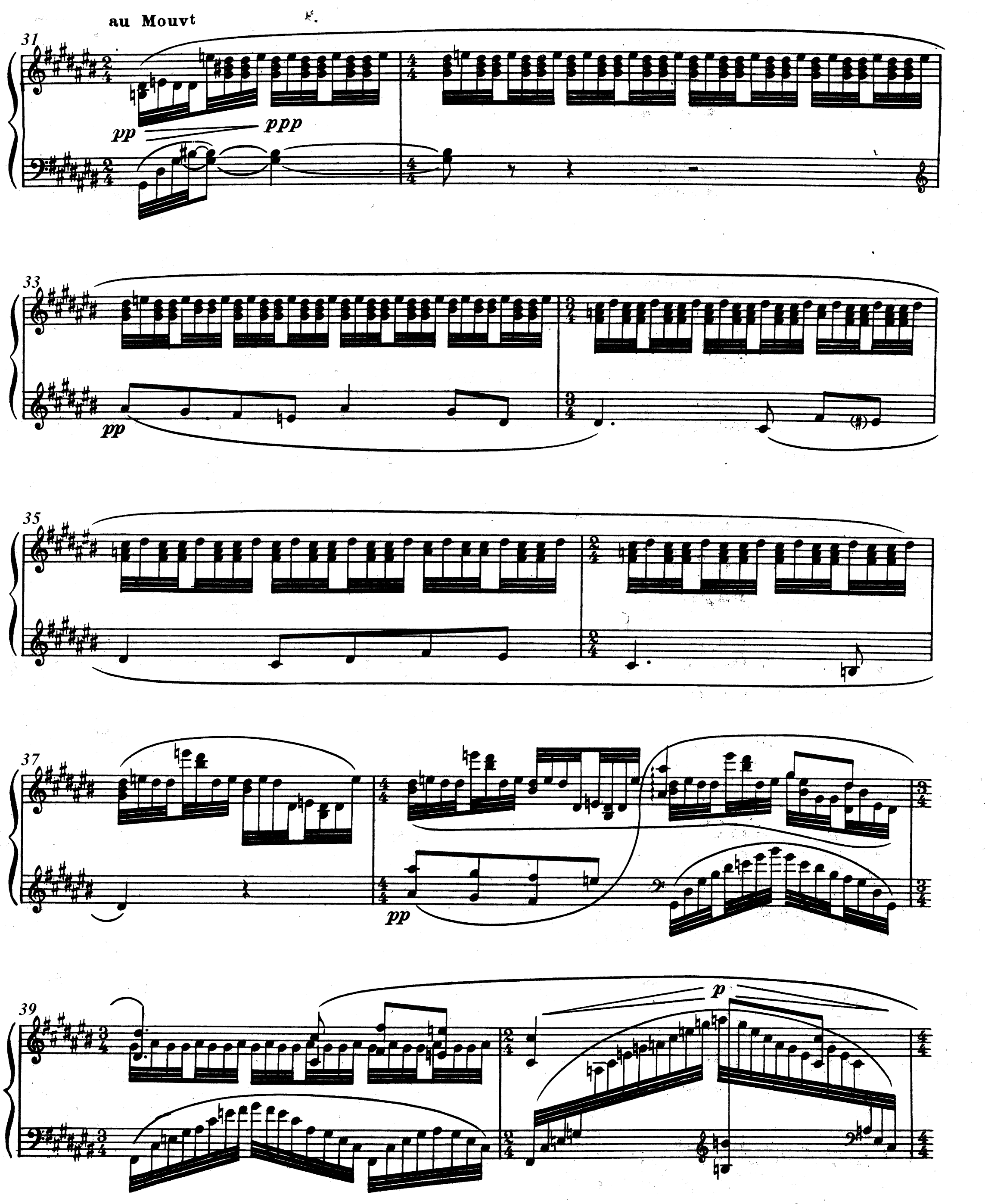


Figure 5.15, continued

Figure 5.15, continued

Figure 5.15, continued

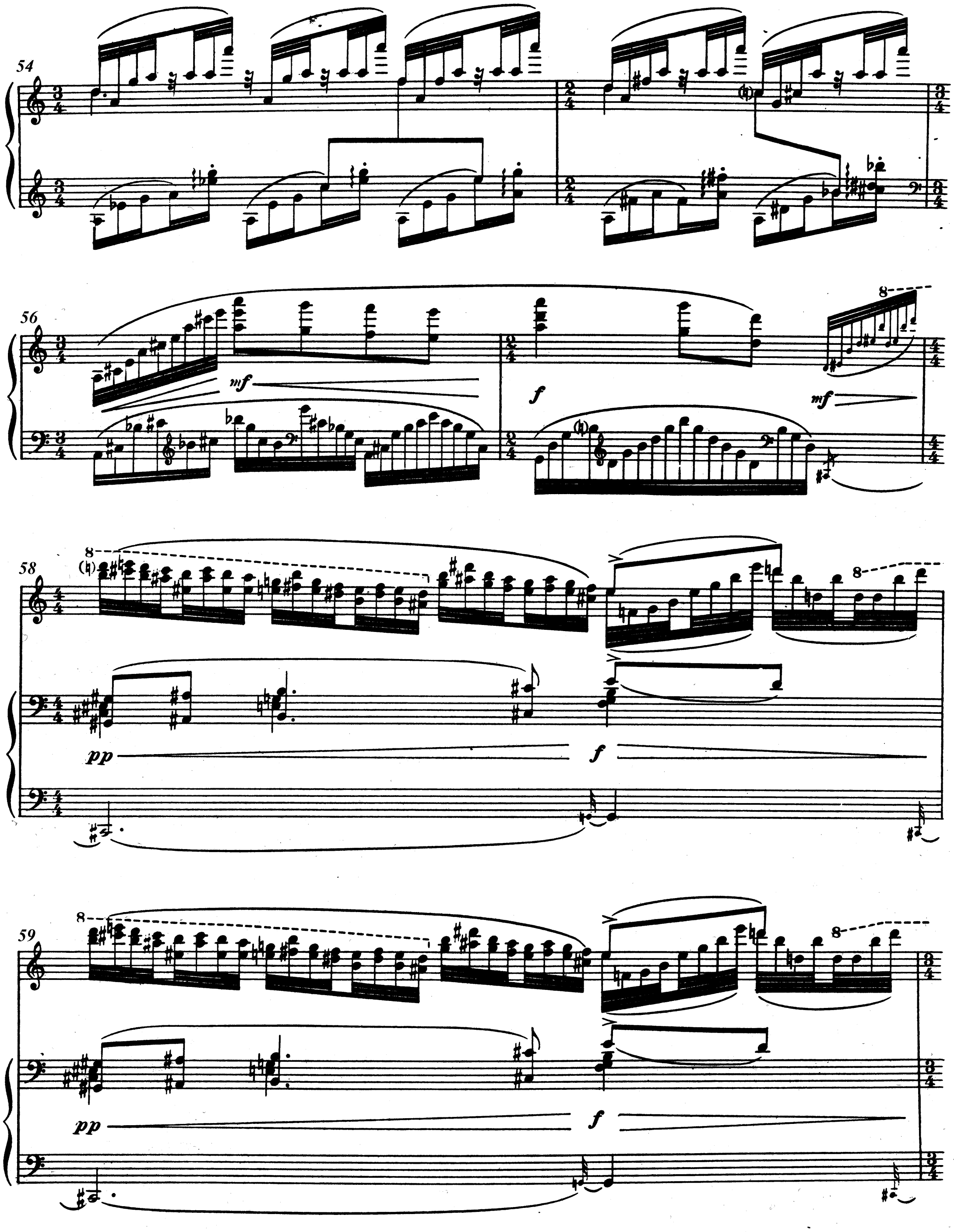


Figure 5.15, continued

Figure 5.15, continued

Figure 5.15, continued

Figure 5.15, continued

Figure 5.15, continued

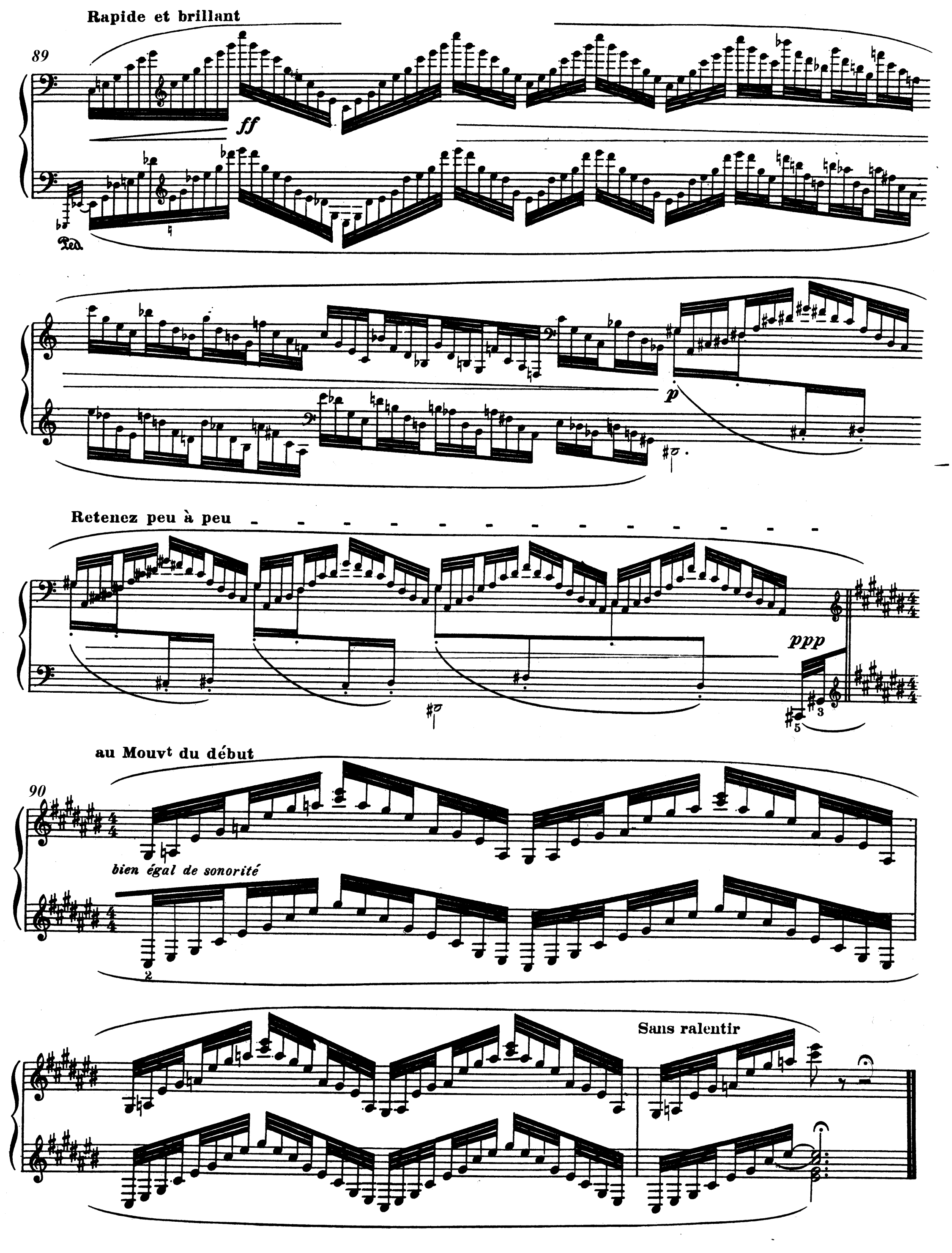


Figure 5.15, continued

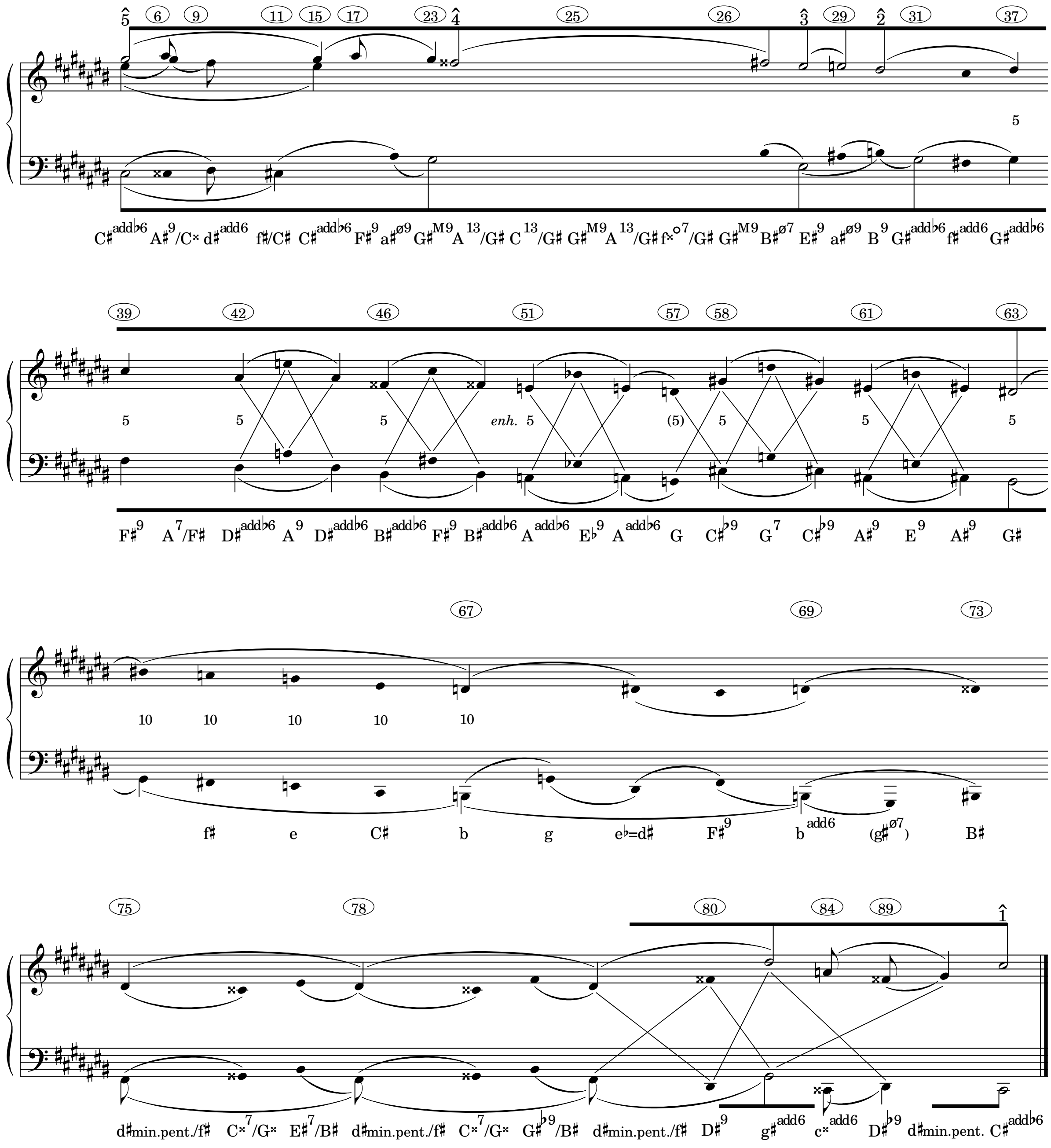


Figure 5.16: Prolongational Sketch of Ravel, *Gaspard de la Nuit*, "Ondine"

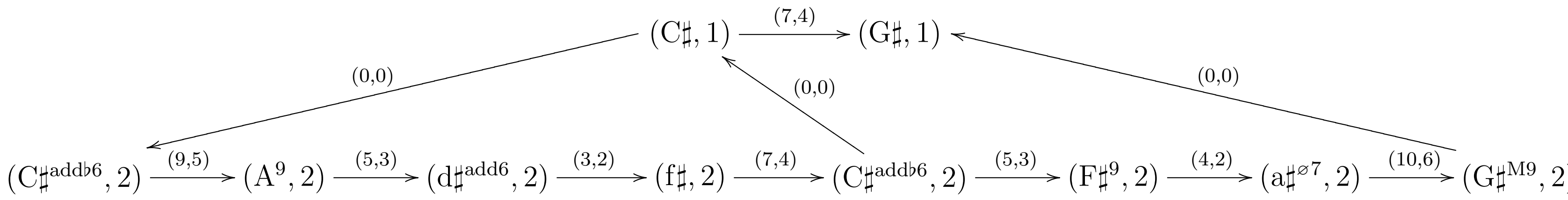


Figure 5.17: Transformational Graph of “Ondine”, mm. 1–23

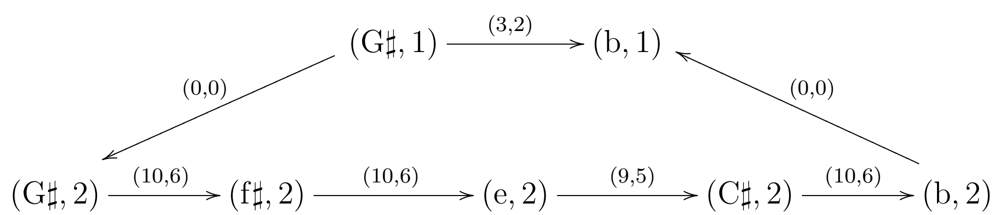


Figure 5.18: Tranformational Graph of “Ondine”, mm. 63–67

Table 5.1: Relative Acoustical Instability in the First Prolongational Span of Figure 5.16

	m. 1	m. 6	m. 9	m. 10.4	m. 11	m. 15
Chord	$C\sharp^{add\flat 6}$	$A\sharp^{9}$	$d\sharp^{add6}$	B^{7}	f♯	$C\sharp^{add\flat 6}$
Prime Limit	5	7	7	7	5	5
	(add6=13)					(add6=13)

CHAPTER 6

CONCLUSIONS

6.1 The Use of Diatonic Theory for Extended Tonal Music

In this dissertation, traditional tonal theories (fundamental-bass theory, tuning theory,[1] diatonic scale theory, and the theory of contrapuntal prolongation) have been combined to form the underpinning for an examination of certain extensions of tonality. We shall now take a large view of this research and summarize how it functions with regard to the post-common-practice repertoire. After providing a synopsis of the theory and its origins, we shall examine more closely the role that the diatonic scale must play in the music in order for this analytical method to be useful. This will allow us to project what other repertoires one may also study in the future using this diatonic model.

As we discovered in Chapter 2, tuning theory and diatonic scale theory are interconnected—even inseparable. Their combination allows us to establish a clear and stable system of scale-degree functions in tonal music. In order to visualize these functions and their musical implementations, Chapter 3 enumerated several graphical means of displaying harmonic relationships in the tonal space that we built in Chapter 2. The *Tonnetz* and just-intonation and diatonic transformational networks offer useful spatial metaphors for tonal function, motion, and prolongation. So that we might use these networks in the analysis of extended-tonal music, Chapter 4 developed tools for supporting analyses of chordal hierarchies. Many other scholars have established contrapuntal analytical methods that complement the largely harmonic-function-based methods given here.[2] In fact, a prolongational analysis would be incomplete without linear analysis. Jones (2002), however, has suggested that it is dangerously easy to select analytically meaningless examples of contrapuntal fluency without first choosing the structural harmonies. For this reason, Chapter 4 provides methods for finding structural harmonies based upon harmonic function and chord construction. Once a preliminary chordal hierarchy is established, linear analysis

[1]Tuning theory may not be a tonal theory itself. It nevertheless forms the basis of certain tonal theories.
[2]Q.v. Section 1.3 and 1.4.

becomes a meaningful way of refining the prolongational analysis and discovering how the transient sonorities arise as elaborations of the structural harmonies.

Although our goal was the analysis of post-common-practice music, all of the tonal theories that we drew upon for this purpose are dependent upon the diatonic basis of tonal music. If the music distorts this diatonic background in particular ways, we can still analyze it from a tonal perspective. Specifically, in Chapter 5 we explored how enharmonic progressions, while causing some "vertigo" at the surface level, may be rectified at a deeper level to a traditional diatonic structure and thus may still function within a prolongational view of the music. Further, we examined how directional tonality may be either subsumed within a prolongational view or emphasized by its transformational graph structure. We also examined how a diatonic background may even inform the analysis of tonal music that explores the limits of nineteenth-century tertian chromaticism. Finally, we tested our tools for asserting prolongational structures within music that features non-functional progressions and non-tertian harmonies.

There is widespread acceptance that the music that we have examined is tonal or at least relies heavily upon tonal constructs. The extended-tonal repertoire, however, sometimes presents seemingly insurmountable obstacles to tonal analysis. Some may argue that Ravel's music is only tangentially related to the tonal musical repertoire. Nevertheless, to the extent that this repertoire relates to traditional tonality, that relationship deserves careful and thorough exploration using the full arsenal of tonal theoretical apparatus. In any music to which a set of popular chord symbols can be applied, even if the progressions do not follow traditional functional paradigms, the chords' relationships may be expressed within the tonal space constructed in this treatise. While this tonal space derives its strength from its diatonic basis, its inherent diatonic bias does impose a limit on its explanatory power. It might be possible to attempt to force some atonal music (e.g. parts of Schoenberg's Op. 11 Piano Pieces) into a diatonic framework,[3] but to the extent that this music relies on post-tonal techniques, our tonal space is powerless to represent those structures.[4] I hope that this theory may, however, enrich the study of a great deal of other post-common-practice music. Mod-7 prolongational networks may serve to illuminate the tonal features of music written by many of the neo-tonal composers of the twentieth century, such as Barber, Bartók, Britten, Copland, Debussy, Ginastera, Hindemith, Martinů, Menotti, Pärt, Persichetti, Prokofiev, Rorem, Shostakovich, Sibelius, Vaughan Williams, and many others.

[3] Schoenberg himself (1931) reharmonized the atonal melody from his Variations for Orchestra, Op. 31, using tonal harmony.

[4] Music that makes clear reference to tonality while also exhibiting structures that are best shown with atonal analysis still requires a combination of analytic techniques. Given that transformational networks may be used to show features of both tonal and atonal music, a unified notation for showing both tonal and atonal structures in the extended-tonal repertoire may be possible.

This analytical method may be extended to the study of other repertoires as well. We have seen that the reference scale given in Table 2.1 privileges modal mixture as a source for chromaticism.[5] In fact, the theory reads any of the traditional modes (except for Locrian) as fundamentally diatonic and analyzable within tonal space. Hence we can include pan-diatonic music among the neo-tonal repertoires that may be examined in tonal space. This inclusion of modal harmony also suggests that it might be possible to use our analytical method to study the non-functional progressions of pre-tonal polyphony.[6] The acceptance of modality within this model of tonal space offers a potential path toward a tonal theory of popular music in the twentieth century as well. These repertoires, however, are beyond the scope of the present study.

6.2 The Place of This Work Within the Field of Music Theory

In this dissertation, we have explored how some older theoretical views of tonality may be revived, reevaluated based on current research, and combined with modern theories in fruitful ways. This work thus joins a growing body of eclectic approaches to analysis.[7] This dissertation is a formalization of my intuitions about tonal and neo-tonal music. While my goal is to build theories objectively using scientific, structuralist, and formalist techniques, all theories are necessarily still based in subjectivity.[8] This subjectivity is not to be seen as a weakness. Just as my view of extended tonal music has been enriched by learning post-tonal analytical approaches to this repertoire, I hope that other musicians will find personal value in a prolongational view of this repertoire.

The successful combination of two or more different theoretical views is nothing new in the field of musicological study. Eschewing one of two theories that appear to disagree, however, is also unfortunately commonplace in past scholarship.[9] For example, it is a lamentable fact that, during the twentieth century, there has been a rift between tuning theory and mainstream music theory. It is problematic that some tuning theorists have accorded a

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[5]Recall that, as a result, both tonicization and modulation require transposition of this scale to the temporary tonic.

[6]Adams's (2005) distinctions between essential and inessential chromaticism in sixteenth-century polyphony are somewhat related to my own differentiation of modal mixture from tonicization and my provisions for respelling certain degrees of the scale in Table 2.1.

[7]Postmodern scholarship has been notoriously accepting of diverse viewpoints. For an intriguing postmodern view of Schenker's theory see Dubiel 1990.

[8]Clifton (1975, 69) discusses the inherent subjectivity of scientific endeavor: "As any true scientist will tell you, scientific objectivity is just as impure as intuitive objectivity. To be sure, it is a valuable ideal, but it is also a human invention, as fallible as it is seductive." See also Brown and Dempster 1989, Brown 1996, Mailman 1996, Sayrs and Proctor 2002, and Childs 2005.

[9]Often two conflicting theories merely focus upon different parameters of the same phenomenon. We have seen that this is certainly true of diatonic theory and tuning theory.

Pythagorean—even religious—significance to the numerology of just intonation's beautiful proportions.[10] Reputable music theorists who work in areas tangential to tuning theory seem to have felt a need to distance their work from the just-intonation camp. One example of this is Carey and Clampitt's (1989) own introductory remarks:[11]

> In the past, [a principled basis for tonal music] was sought in the physical phenomenon of the overtone series. This approach was found wanting in important respects: not only did the overtone hypothesis fail to generalize to non-triadic music, but it also inadequately and inconsistently explained features within the major-minor tonal system, such as the status of the minor triad as a consonance and as functionally equivalent to the major triad.[12] In recent years, diatonic set theory has provided an alternative perspective, which generally has proceeded from the assumption of an ideal equal division of the octave.

While Carey and Clampitt implicitly acknowledge the possibility that tuning theory may also have some explanatory power, they cite no resources that defend tuning theory against the critical appraisals to which they refer the reader. This is perhaps because in recent years tuning theory has progressed apart from academic music theory.[13] Many theorists in the twentieth century have been justifiably critical of the explanatory power of the harmonic series, but the effective removal of the field of musical tuning from the mainstream in music theory is unfortunate and perhaps also detrimental to musical scholarship. Theorists should indeed learn from the failings of past theories based upon just intonation, thus taking care not to invoke unsubstantiated numerological and theoretical ideals in support of their assertions and analyses. Ideally the development and revision of theories comes most profitably from their exchange and from discussion of their points of disagreement.

It is my hope that the relationship between diatonic theory and tuning theory introduced here may form part of a renewed desire in the music theory community to recombine newer theories with discarded historical ideas in enlightening ways. In addition to tuning theory and diatonic theory, this dissertation has brought together other theories as well. We have paired transformational analysis with hermeneutics (in Section 5.1), dualist function theory (based on Harrison 1994) with linear analysis (in Chapter 4), acoustics with root analysis (in

[10]To be fair, just-intonation enthusiasts are not the only theorists who have had an unhealthy fancy for numerology. See, for instance, Clark 1999. For an intriguing and entertaining discussion of numerology as a symptom of a form of innumeracy (mathematical illiteracy), see Paulos 1988.

[11]Recall that in Section 2.2 we found Agmon (1989) also separating his work from tuning theory.

[12]Carey and Clampitt's footnote: "The overtone hypothesis has been discussed at length elsewhere. We refer the reader to critical appraisals in Babbitt 1972a and 1972b, Cogan and Escot 1976, 139–141, and Lerdahl and Jackendoff 1983, 290–293."

[13]The one English-language theoretical journal devoted to tuning is *Xenharmonikôn*, but there is a very active internet community of tuning theorists.

Section 4.1), and number, matrix, and group theory with scale-degree function and diatonic spelling (in Chapter 2). There is a seemingly limitless ways of combining ideas from the many different subfields of music-theoretical thought. I hope that this study and any future works that it inspires are successful in combining established scientific, analytical, and rhetorical techniques to further challenge the taxonomies that have historically subdivided our field.

APPENDIX A

GLOSSARY OF MATHEMATICAL TERMS AND SYMBOLS

$\mathbb{P}$ is the **set** of **prime numbers**.[1]

$\mathbb{Q}$ is the **set** of **rational numbers**.

$\mathbb{R}$ is the **set** of **real numbers**.

$\mathbb{Z}$ is the **set** of **integers**.

$\boldsymbol{x} \subset \boldsymbol{y}$ indicates that the **set** x is a subset of the **set** y. See discussion at **set**.

$\boldsymbol{x} \in \boldsymbol{y}$ means that the object or number x is an **element** of the **set** y. See discussion at **set**.

$|\boldsymbol{x}|$ is an abbreviation for the **absolute value function**.

$\lfloor \boldsymbol{x} \rfloor$ represents the floor or trunc function, which returns the greatest **integer** less than or equal to x.

$\lceil \boldsymbol{x} \rceil$ symbolizes the ceiling function, which returns the smallest **integer** greater than or equal to x.

$\lfloor \boldsymbol{x} \rceil$ denotes the nearest-integer (nint) or **rounding** function, which returns the smallest **integer** greater than or equal to x for x with fractional part $> \frac{1}{2}$ and the greatest integer less than or equal to x for x with fractional part $< \frac{1}{2}$. For x with fractional part $= \frac{1}{2}$, different mathematical applications may require rounding up or down. In this dissertation, when x has fractional part $= \frac{1}{2}$, the function is defined to give the integer y with smallest **absolute value** such that $|y| > |x|$.

$\boldsymbol{\{x, y, z\}}$ indicates that the items or expressions enclosed in curly brackets form or define a **set**.

[1] Many of my glossary entries are drawn from the glossary in Hook 2002. A few have also been drawn from the wonderful resource <http://mathworld.wolfram.com/>.

abelian group This is another name for **commutative group**.

absolute value The absolute value function on a number x, commonly designated $|x|$, returns x if $x \geq 0$ and $-x$ if $x < 0$. This allows for comparable measurements of distance away from 0 in both the positive and negative directions.

additive group Any **group** where the **group operation** is ordinary addition, or possibly addition **mod** n, may be called an additive group.

associative A **binary operation** $\otimes$ on a set S is associative if the equation $(x \otimes y) \otimes z = x \otimes (y \otimes z)$ holds for all x, y, and z in S. If $\otimes$ is associative, then we may simply write $x \otimes y \otimes z$ (without parentheses) with no danger of ambiguity. A **group operation** is required to be associative by definition.

bijective A **function** is bijective if it is both **injective** (one-to-one) and **surjective** (onto).

binary operation If S is a **set**, a binary operation on S is a means of combining two **elements** of S to produce an **element** of S. Familiar examples of binary operations include the operations of addition and multiplication on the set of real numbers. In this dissertation, we shall use the conventional symbols for the familiar operations of addition, multiplication, division, etc.; and we shall represent a binary operation on two elements of a set (where the binary operation on the set has already been defined) by juxtaposing two variables representing elements of a set (e.g. xy).

cardinality The number of distinct **elements** in a finite **set** S is called the cardinality of S.

column vector A column vector is a $m \times 1$ **matrix**—that is, a matrix consisting of a single column of values.

commutative If a **binary operation** $\otimes$ is defined on a **set** S, and if x and y are any two **elements** of S, then x and y are said to commute if their **product** does not depend on order—that is, $x \otimes y = y \otimes x$ must hold for all x and y in S. A **group operation** is not required to be commutative.

commutative group If a **set** can be shown to be a **group**, and its **group operation** satisfies the conditions of the **commuative property** (i.e. the product of two group **elements** does not depend on their order), then the group may be called a commutative group.

component A single object in an **ordered n-tuple** is called a component.

determinant For every sqare $(n \times n)$ **matrix**, a value called the determinant of the matrix may be calculated. For a 1×1 matrix, the determinant is simply the unique entry in the matrix. For a 2×2 matrix

$$\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix},$$

the determinant is $a_1 \cdot b_2 - a_2 \cdot b_1$; and for a 3×3 matrix,

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix},$$

the determinant is

$$a_1 \cdot b_2 \cdot c_3 - a_1 \cdot b_3 \cdot c_2 + a_2 \cdot b_3 \cdot c_1 - a_2 \cdot b_1 \cdot c_3 + a_3 \cdot b_1 \cdot c_2 - a_3 \cdot b_2 \cdot c_1.$$

Beyond the size 3×3, a matrix's determinant is typically derived recursively from its submatrices. Matrices that have determinant 0 are said to be singular, and matrices that have determinant 1 are said to be unimodular.

element One member of a **set** or **group** is called an element and can be denoted $a \in A$, where a is the element, and A is the set.

function If A and B are **sets**, a function f from A to B may be thought of, informally, as a sort of abstract computing machine into which one feeds an input x in set A and from which emerges the output y in set B. More rigorously, the function f is identified with the set of all ordered pairs (x, y) that satisfy the relationship defined by f. The essential point of the definition is that the function determines one and only one y in B for each x in A; it is possible, however, that two different xs may produce the same y, or that some ys may not appear in association with any xs. (For related definitions, see **injective** and **surjective**.) The unique y associated with the **element** x is conventionally denoted $f(x)$ (pronounced "f of x"), signifying the value of the function f applied to the element x.

group If G is a **set** and $\otimes$ is a **binary operation** on G, then G is said to form a group if the following three conditions are satisfied:

(1) the operation $\otimes$ is **associative**; that is, $(x \otimes y) \otimes z = x \otimes (y \otimes z)$ for all x, y, and z in G;

(2) there exists an **identity element**, an **element** of G such that $x \otimes e = e \otimes x = x$ for every x in G; and

(3) every **element** x of G has an **inverse** in G, an element x^{-1} such that $x \otimes x^{-1} = x^{-1} \otimes x = e$.

A group operation is not required to be **commutative**, but it may be. A general group operation is often represented using multiplicative notation, and may even be called "multiplication", even if it bears little resemblance to the familiar multiplication of numbers. The study of groups and their properties is the aim of group theory.

group operation The **binary operation** in the definition of a **group** is referred to as its group operation.

homomorphism, homomorphic In general, a homomorphism is a **function** from one mathematical structure to another of the same type (group, graph, etc.) that preserves important structural relationships among the objects being mapped, whatever those relationships may be. A function f from a **group** G to a group H is a homomorphism if it preserves group products; that is, whenever x, y, and z are **elements** of G such that $xy = z$, then the equation $f(x)f(y) = f(z)$ holds in H.

identity element If a **binary operation** $\otimes$ is defined on a set S, then an **element** e of S is called an identity element if, for every $x \in S$, $x \otimes e = e \otimes x = x$. For example, 0 is the identity element for the operation addition in the set of real numbers, and 1 is the identity element for multiplication. A **group** is required to have an identity element by definition.

iff The word "if" with two "f"s is an abbreviation for "if and only if". In mathematical terms, an "if p then q" statement means that q is true whenever p is true, but possibly also under other circumstances. An "iff p then q" statement, however, indicates that p must be true for q to be true as well; otherwise q must be false.

injective A **function** f from a set A to a set B is injective (or one-to-one) if it always maps distinct **elements** of A to distinct **elements** of B; that is, the only way the equation $f(x) = f(y)$ can hold is if $x = y$. If A and B are finite sets, an injective function from A into B can exist only if the **cardinality** of A is less than or equal to the **cardinality** of B.

integers The **set** of integers, $\mathbb{Z}$, includes the positive and negative whole numbers and zero $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$.

inverse If $\otimes$ is a **binary operation** on a set S, and the **element** e is an **identity element** of S, and x is any element of S, then the element y in S may be called the inverse of x if $x \otimes y = y \otimes x = e$. The inverse of x is often denoted x^{-1}. Elements of a **group** are required to have inverses by definition.

isomorphism, isomorphic In general, an isomorphism is a **homomorphic function** from one mathematical structure to another that is **bijective** (one-to-one and onto). In other words, the function creates a perfect one-to-one correspondence between the elements of the two structures which moreover preserves all significant structural relationships among those elements. Two mathematical structures are called isomorphic if there exists an isomorphism mapping from one to the other. This means, intuitively, that the structures are identical in all significant structural ways (although the elements themselves may be different objects).

map, mapping A map is simply a **function**.

matrix Any rectangular array of numbers or other mathematical objects may be called a matrix. A matrix arranged in m rows and n columns is called an $m \times n$ matrix. (Examples of 2×2 and 3×3 matrices can be found in Section 2.3.) The value in the i^{th} row and j^{th} column of a matrix A is typically denoted a_{ij}. If A is an $m \times n$ matrix, and B is an $n \times p$ matrix (for the same n), then the product AB is the $m \times p$ matrix defined by the equation $(ab)_{ij} = A_{i1} \cdot b_{1j} + a_{i2} \cdot b_{2j} + \cdots + a_{in} \cdot b_{nj}$; that is, the component in the i^{th} row and j^{th} column of AB is obtained by multiplying each value in the i^{th} row of A by the corresponding value in the j^{th} column of B and adding the results. For square matrices ($n \times n$), a value called the **determinant** of the matrix may be calculated. When the determinant is 1, the matrix is said to be unimodular.

mod, modulo The operation $c = a \bmod b$ returns the remainder or common residue of $\frac{a}{b}$, defined to be a nonnegative number smaller than b such that $\frac{a-c}{b} \in \mathbb{Z}$. The set of integers mod n forms an **additive group** of order n, and is commonly designated $\mathbb{Z}_n$. Many everyday measurement systems use modular addition over $\mathbb{Z}_n$. For example, clocks use mod-12 arithmetic, so that five hours past 10:00 in the evening is 3:00 in the morning ($(10 + 5) \bmod 12 = 3$).

ordered pair An ordered pair is a kind of list of two objects. Any two objects x and y (they can be numbers, elements of some set, or even objects of entirely different types) can form an ordered pair (x, y), where x is the first component (or coordinate) and y is the second component. Two ordered pairs (x_1, y_1) and (x_2, y_2) are equal if and only if both $x_1 = x_2$ and $y_1 = y_2$. Note also that, because they are "ordered", $(1, 2)$ is different from $(2, 1)$. Ordered pairs are usually enclosed in parentheses, but other notations may be useful as well.

ordered triple, ordered *n*-tuple By extension of the concept of **ordered pair**, we may form ordered triples (x, y, z), ordered 4-tuples or quadruples (x, y, z, w), and larger n-tuples.

prime number, primes A prime number (or prime integer, often simply called a "prime" for short) is a positive integer $p > 1$ that has no positive integer divisors other than 1 and p itself. (More concisely, a prime number p is a positive integer having exactly one positive divisor other than 1.) For example, the only divisors of 13 are 1 and 13, making 13 a prime number, while the number 24 has divisors 1, 2, 3, 4, 6, 8, 12, and 24 (corresponding to the prime factorization $24 = 2^3 \cdot 3^1$), making 24 not a prime number. The set of all primes is often denoted $\mathbb{P}$.

prime limit Any **rational number** $q = \frac{a}{b}$ in lowest terms, where a and $b \in \mathbb{Z}$, has prime limit p iff a's factors and b's factors $\subset \{\text{primes} \leq p\}$. In other words, if a rational number q has a particular prime limit, all prime factors of both the numerator and denominator of q in lowest terms are less than or equal to the prime limit. Those prime factors that combine in various ways to form all members of the set of rationals with prime limit p are called the set's prime generators.

product While a product is normally the arithmetic result of ordinary multiplication, in **group** theory, the product is the result of applying the **group operation** to two **elements** of the group.

rationals The **set** of rational numbers, $\mathbb{Q}$, includes all numbers that are expressible as the quotient of two integers. ($q \in \mathbb{Q}$ iff $q = \frac{a}{b}$ where a and $b \in \mathbb{Z}$.) Note that the **integers** are a subset of the rationals. ($\mathbb{Z} \subset \mathbb{Q}$.)

real numbers The **set** of real numbers, $\mathbb{R}$, includes all numbers both rational and irrational, but no imaginary numbers. Note that the **integers** and the **rational numbers** are subsets of the real numbers.

reciprocal matrix Also known as inverse matrices, reciprocal matrices are square $n \times n$ **matrices** A and A^{-1} such that the product of $A \cdot A^{-1} = I$, where I is the identity matrix. The identity matrix is a square matrix of with the value 0 in all components except for the components along the diagonal $i_{0,0}, i_{1,1}, i_{2,2}, \ldots, i_{n,n}$, each of which contains the value 1. See discussion at **matrix** for more on matrix components and matrix multiplication.

rounding Rounding is the process of approximating a quantity for the sake of convenience or necessity. The rounding of a rational or real number x to the nearest integer is given by the nearest integer function, symbolized $\lfloor x \rceil$. For x with fractional part $= \frac{1}{2}$, two integers are equally proximate to x. In this dissertation, when x has fractional part $= \frac{1}{2}$, the nearest integer function is defined to give the integer y with smallest **absolute value** such that $|y| > |x|$.

row vector A row **vector** is a $1 \times n$ **matrix**—that is, a matrix consisting of a single row of values.

set A set is a collection of objects in which order has no significance, and multiplicity is generally also ignored. Members of a set are often referred to as **elements** and the notation $a \in A$ is used to denote that a is an element of a set A. Further, the notation $A \subset B$ is often used to indicate that A is a subset of B, which means that every element of set A is also an element of set B. The study of sets and their properties is the object of set theory.

subgroup A subset H of a **group** G is called a subgroup of G if H itself forms a group with respect to the same group operation defined in G. To show that a subset H forms a subgroup of G, one must verify the following three conditions:

(1) H contains the **identity element** of G;

(2) H is closed under products—that is, if x and y are elements of H, then xy must also be an element of H.

(3) H is closed under inverses—that is, if x is an element of H, then x^{-1} must also be an element of H.

surjective A **function** f from a set A to a set B is surjective (or onto) if for every y in B there exists some x in A such that $f(x) = y$. If A and B are finite sets, a surjective function from A onto B can exist only if the **cardinality** of A is greater than or equal to the **cardinality** of B.

vector Music theorists use the term simply to mean **ordered *n*-tuple**. (E.g., the interval-class vector is an ordered sextuple.) In some circumstances it will be valuable to represent a vector in the form of a matrix as a **row vector**.

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BIOGRAPHICAL SKETCH

Robert Tyler Kelley

Robert Kelley was born on June 5, 1976, in Arlington, Virginia. He began music study as a child, and learned to play the piano and saxophone. Upon entering Furman University as a scholarship student, Robert decided to divide his interests between piano performance, music theory, and composition, while maintaining his skills on the saxophone in the Furman band program. At Furman, Robert studied keyboard with Derek Parsons, Charles Tompkins, and Daniel Koppelman, and composition with Mark Kilstofte and Daniel Koppelman. In 1997 he received the Bell Tower Scholarship to play the university's carillon for special occasions.

After receiving his B.M. degree from Furman University in 1998, Robert entered the M.M. program in piano performance and music theory/composition at James Madison University. At JMU, Robert taught class piano as a teaching assistant and an instructor, taught private piano lessons, and studied piano with Eric Ruple and composition with John Hilliard. In 2000, Robert received the James Riley Memorial Composition Award for his *Four Songs on Buddhist Texts*.

In January 2001, Robert began his Ph.D. work in music theory at the Florida State University, studying voice leading and prolongation with Evan Jones, the semiotics of opera with Matthew Shaftel, and Schenkerian analysis with advisor Michael Buchler. He received both research and teaching assistantships, serving as a primary instructor for basic theory and ear training courses. Robert has served as president of the FSU Music Theory Society, as the organization's webmaster, and as program chair for the society's annual Theory Forum. In 2003, he received the award for best student paper at the annual meeting of Music Theory Southeast. Robert has also presented talks at other conferences including the second annual meeting of the Music Theory Society of the Mid-Atlantic and symposia on the music of Hugo Wolf and on Theatre & Music in London circa 1700.

As a performer, Robert is an active accompanist specializing in harpsichord continuo and new music. In his spare time, Robert enjoys the company of his family and friends, world cuisine, cinema, hiking, puzzles, strategy games, and punning.